The Finite-Symbol Embedding Theorem: Phase Space Reconstruction for Finite Symbolic Dynamical Systems

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Abstract

Takens' embedding theorem provides conditions under which the attractor of a smooth dynamical system may be reconstructed from scalar observations via delay coordinates. However, many systems of interest—particularly symbolic, arithmetic, and computational processes—do not satisfy the smoothness and diffeomorphic assumptions required by the classical theorem.

In this work, we propose a finite-symbol extension of delay embedding, replacing the requirement of smooth equivalence with finite, measurement-based structural consistency. We define a framework for phase space reconstruction in finite symbolic systems and introduce the Finite-symbol Embedding Principle, which asserts that attractor geometry may be recovered within bounded uncertainty from delay embeddings of symbolic trajectories.

This formulation aligns with a measurement-grounded view of mathematics, in which all computation occurs within finite symbolic systems and all results carry bounded uncertainty. The proposed framework provides a pathway to analyze discrete and symbolic dynamical systems—such as integer iteration processes—using geometric methods traditionally reserved for continuous systems.

1 Introduction

Classical dynamical systems theory operates within a framework of smooth manifolds and continuous transformations. Within this context, Takens' theorem establishes that delay-coordinate embeddings can reconstruct the geometry of an attractor from scalar observations.

However, many systems of contemporary interest—integer sequences, symbolic processes, and computational algorithms—are inherently discrete and finite. These systems do not admit smooth structure, yet they exhibit rich dynamical behavior, including apparent attractors, recurrence, and structured trajectories.

This motivates the following question:

> Can the principle of phase space reconstruction be extended to finite symbolic systems, where smoothness is absent but structure remains observable?

We propose that such an extension is not only possible, but natural when mathematics is grounded in finite measurement and symbolic computation.

2 Classical Takens Framework

Let M be a compact manifold of dimension d, $\phi : M \to M$ a diffeomorphism, and $h : M \to \mathbb{R}$ a smooth observation function.

The delay embedding map is defined as:

$$\Phi(x) = (h(x), h(\phi(x)), \ldots, h(\phi^{m-1}(x)))$$

Takens' theorem states that for generic (ϕ, h) and $m > 2d$, Φ is an embedding, preserving the topology of the attractor.

This result relies on:

- Smoothness of ϕ and h
- Existence of an underlying manifold
- Infinite precision

These assumptions do not hold for symbolic or computational systems.

3 Geofinite Framework

We adopt a Geofinite perspective:

Definition 1 (Finite Symbolic System). *A finite symbolic dynamical system is a pair* $(\mathcal{S}, T)$ *where:*

- $\mathcal{S}$ *is a finite or finitely representable symbolic space*
- $T : \mathcal{S} \to \mathcal{S}$ *is an update rule defined by symbolic operations*

Definition 2 (Trajectory). *Given* $x_0 \in \mathcal{S}$*, a trajectory is the sequence:*

$$x_k = T^k(x_0)$$

Definition 3 (Measurement Constraint). *Each observation* x_k *is represented with finite precision and bounded uncertainty:*

$$x_k \in B_\epsilon(\tilde{x}_k)$$

where B_ϵ *is a finite uncertainty region.*

4 Geofinite Delay Embedding

We define the delay embedding:

$$X_k = (x_k, x_{k-\tau}, x_{k-2\tau}, \ldots, x_{k-(m-1)\tau})$$

with:

- embedding dimension m
- delay τ

Remark 1. *Unlike the classical case,* X_k *is not assumed to lie on a smooth manifold. Instead, it defines a trajectory in a finite-dimensional measurement space with bounded resolution.*

5 Geofinite Embedding Principle

Proposition 1 (Finite Structural Recoverability). *Let $(\mathcal{S}, T)$ be a finite symbolic system generating trajectories $\{x_k\}$. Then, for sufficiently large embedding dimension m and trajectory length N, the delay embedding $\{X_k\}$ reconstructs a geometric structure that is consistent with the underlying dynamical process within measurement bounds.*

Theorem 1 (Geofinite Embedding Principle). *Given a finite symbolic dynamical system with non-degenerate evolution and bounded measurement uncertainty, there exists an embedding dimension m^* such that the delay embedding Φ_m produces a phase-space representation whose attractor structure is stable under finite perturbations:*

$$d(\mathcal{A}_m, \mathcal{A}_{m+\Delta m}) < \epsilon$$

for sufficiently large $m \geq m^*$.

Remark 2. *This replaces diffeomorphic equivalence with* geometric stability under measurement constraints.

6 Interpretation

Under this framework:

- Attractors are finite geometric structures
- Convergence is geometric contraction
- Cycles are closed trajectories
- Divergence corresponds to bounded excursions within a finite region

7 Implications

7.1 Discrete Systems

Systems such as integer iteration processes may be analyzed as dynamical systems without requiring continuous structure.

7.2 Measurement-Based Mathematics

All results are inherently:

- finite
- bounded
- dependent on representation

7.3 Extension of Classical Theory

This framework suggests that Takens' theorem may be viewed as a special case of a broader principle:

> Phase space reconstruction is a property of information-rich trajectories, not solely of smooth systems.

8 Experimental Pathway

To validate the framework:

1. Generate symbolic trajectories
2. Construct delay embeddings
3. Analyze geometric structure
4. Test stability under:
   - embedding dimension changes
   - noise / perturbation

9 Conclusion

We propose a GEOFINITE extension of delay embedding that applies to finite symbolic systems. By replacing smooth equivalence with measurement-based geometric stability, we extend the applicability of phase space reconstruction to a broader class of systems.

This provides a foundation for analyzing discrete mathematical processes as dynamical systems, bridging symbolic computation and geometric analysis.