# **The Attralucian Essays**:

Exploring the Finite

First Edition

 Typeset in LaTeX

# The Attralucian Essays

*Finite Symbolic Mechanics*: On Quaternions

Kevin R. Haylett

# Chapter 1

# Quaternions as Ordered Rotational Reconstruction

## Overview

In the preceding essay, complex numbers were reinterpreted as dynamical reconstructions arising from delay-coordinate embeddings of measured signals. Their apparent imaginary structure was shown to encode phase relationships inherent in temporal measurement, rather than to represent an independent ontological dimension.

The present chapter extends this program to quaternions. Historically introduced by William Rowan Hamilton as a four-dimensional extension of complex numbers, quaternions are typically presented as algebraic objects inhabiting a higher-dimensional space. Their non-commutative multiplication is treated as a mathematical curiosity, and their effectiveness in describing three-dimensional rotation is often taken as evidence of deeper geometric structure.

Within the framework of Finite Symbolic Mechanics (FSM), a different interpretation becomes available. Quaternions are not extensions into higher-dimensional space, but minimal symbolic containers required to preserve ordered rotational transformations under finite measurement constraints.

The central claim of this chapter is:

*Quaternions arise as the minimal admissible symbolic structure that preserves magnitude, invertibility, and ordered composition of three-dimensional rotations.*

Their non-commutativity is not an abstract algebraic property. It is the direct symbolic trace of ordered process.

# Rotations as Finite Transformations

We begin from an FSM admissible position.

A spatial configuration is represented as a measured vector:

$$\mathbf{v} = (x, y, z),$$

where each component is a measured number with finite resolution and uncertainty.

A rotation is not a static object, but a transformation:

$$R : \mathbf{v} \mapsto \mathbf{v}'.$$

This transformation must preserve measurable structure, in particular magnitude:

$$|\mathbf{v}'| = |\mathbf{v}|,$$

within finite tolerance.

We therefore state:

> **Axiom 1:** All physically admissible rotations are finite transformations preserving measurable magnitude.

# Composition and Closure

If rotations are to form a usable symbolic system, their composition must remain within the same class:

$$R_2 \circ R_1 = R_3.$$

This implies closure under composition.

> **Axiom 2:** The composition of admissible rotations must remain admissible.

However, in three-dimensional space, composition is not commutative:

$$R_2 \circ R_1 \neq R_1 \circ R_2.$$

This is not a mathematical inconvenience. It is a measured property of spatial interaction.

> **Axiom 3:** Ordered composition of rotations is physically distinct and must be preserved symbolically.

## Failure of Three-Component Representations

A natural attempt is to represent rotations using three-component objects, corresponding to spatial axes. However, such representations—most notably Euler angles—fail to preserve all required properties simultaneously.

In particular:

- Ordered composition leads to degeneracies (gimbal lock)
- Distinct rotational histories may collapse to identical symbolic states
- Invertibility and continuity are not globally preserved

From an FSM perspective, this constitutes a failure of admissibility:

*The symbolic container is insufficient to preserve distin-*

*guishability under finite composition.*

## Minimal Extension: The Quaternion Container

To preserve magnitude, invertibility, closure, and ordered composition, an additional component is required.

We therefore introduce a rotational container of the form:

$$Q = (s, \mathbf{u}),$$

where:

- $s$ is a scalar component
- $\mathbf{u} = (x, y, z)$ is a vector component

This yields the familiar quaternion form:

$$q = s + xi + yj + zk.$$

Within FSM, this is not interpreted as a four-dimensional spatial object. Rather:

*The scalar component encodes rotational phase, while the vector component encodes rotational axis.*

The fourth term is not an additional dimension. It is the minimal symbolic cost required to preserve ordered rotational structure.

**FSM Insight**

The scalar component is not a fourth *spatial* coordinate; it is the minimal bookkeeping device required to keep the ordered sequence of axis alignments from collapsing under finite composition.

## Derivation of the Multiplication Rule

Let:

$$Q_1 = (s_1, \mathbf{u}_1), \qquad Q_2 = (s_2, \mathbf{u}_2).$$

Their composition must yield a new admissible rotation:

$$Q_3 = Q_2 Q_1.$$

To preserve magnitude and relational structure, the resulting components must incorporate:

- Alignment between axes: captured by a dot product
- Oriented ordering between axes: captured by a cross product

This forces the multiplication rule:

$$(s_2, \mathbf{u}_2)(s_1, \mathbf{u}_1) = (s_2 s_1 - \mathbf{u}_2 \cdot \mathbf{u}_1,\ s_2 \mathbf{u}_1 + s_1 \mathbf{u}_2 + \mathbf{u}_2 \times \mathbf{u}_1).$$

This is precisely the quaternion product.

## Non-Commutativity as Process Trace

The cross product satisfies:

$$\mathbf{u}_2 \times \mathbf{u}_1 = -(\mathbf{u}_1 \times \mathbf{u}_2),$$

which immediately implies:

$$Q_2 Q_1 \neq Q_1 Q_2.$$

Thus:

*Non-commutativity arises directly from the preservation of ordered rotational process.*

It is not an algebraic anomaly. It is a necessary condition for representing finite transformation history.

## Rotation as a Closed Transformation Process

The standard quaternion rotation:

$$\mathbf{v}' = q\mathbf{v}q^{-1}$$

can now be reinterpreted.

Rather than an abstract algebraic identity, it represents a closed transformation loop:

- encode the rotational state
- apply transformation
- return to the measurable frame

This ensures:

- preservation of magnitude
- preservation of orientation
- closure under composition

## Gimbal Lock Revisited

In classical treatments, gimbal lock is described as a singularity of Euler angle coordinates.

Within FSM, a deeper interpretation is available:

*Gimbal lock is a loss of distinguishability in rotational process space due to an insufficient symbolic representation.*

Distinct ordered transformations collapse into identical coordinate states. The representation fails to preserve process history.

Quaternions avoid this failure because they encode rotation as a unified process container rather than as a sequence of independent coordinate operations.

# Quaternion Admissibility

We may now state the central result.

**Proposition (Quaternion Admissibility).**

*A quaternion is the minimal finite symbolic container that preserves magnitude, invertibility, and ordered composition of three-dimensional rotations under finite measurement constraints.*

# Mathematics as Process Compression

This result aligns directly with the broader FSM program.

- Complex numbers compress phase relations arising from delay reconstruction
- Quaternions compress ordered rotational transformations

In both cases, algebra is not primary. It is a compression of underlying finite processes.

# Quaternion Phase Portraits: Making Ordered Rotation Visible

## Overview

The preceding sections established that quaternions act as finite symbolic containers preserving ordered rotational transformations. However, much of this structure remains hidden when quaternions are treated purely algebraically.

This section introduces a constructive perspective: by evolving a quaternion as a function of a parameter (typically time), we generate explicit trajectories on the space of orientations. These trajectories may be visualised as phase portraits on the unit sphere.

In doing so, the implicit temporal and ordered structure encoded within quaternion algebra becomes directly observable.

## Time-Dependent Rotational Operators

Let a rotation be parameterised by a finite variable $t$, representing either physical time or an abstract progression parameter.

A unit quaternion may then be written as:

$$q(t) = \cos\left(\frac{\theta(t)}{2}\right) + \sin\left(\frac{\theta(t)}{2}\right)(u_x i + u_y j + u_z k),$$

where:

- $\theta(t)$ is a finite, measurable rotation angle,
- $(u_x, u_y, u_z)$ is a unit vector defining the rotation axis.

Given an initial vector $\mathbf{v}_0$, its evolution under the rotation is:

$$\mathbf{v}(t) = q(t)\,\mathbf{v}_0\,q(t)^{-1}.$$

This defines a trajectory on the unit sphere.

## Phase Portraits on the Sphere

The path traced by $\mathbf{v}(t)$ constitutes a phase portrait of the rotational process.

Several important cases arise:

- **Single-axis rotation:**
  For $\theta(t) = \omega t$ and fixed axis $\mathbf{u}$, the trajectory forms a great circle on the sphere.
- **Multi-axis ordered rotation:**
  Consider a composition:

  $$q(t) = q_z(\omega_z t)\,q_y(\omega_y t)\,q_x(\omega_x t).$$

  The resulting trajectory is no longer a simple circle, but a structured path reflecting ordered composition.

- **Non-commutative variation:**
  Changing the order of composition:

  $$q'(t) = q_x(\omega_x t)\, q_y(\omega_y t)\, q_z(\omega_z t),$$

  produces a distinct trajectory, despite identical component rotations.

## Ordering as Observable Structure

The distinction between:

$$q_z q_y q_x \quad \text{and} \quad q_x q_y q_z$$

is not merely algebraic. It produces different phase portraits.

Thus:

*The non-commutativity of quaternion multiplication becomes directly observable as divergence in trajectory space.*

This provides a concrete interpretation of ordered composition:

- The algebra encodes process ordering
- The phase portrait reveals it geometrically

## FSM Interpretation

Within Finite Symbolic Mechanics, this construction carries a clear interpretation:

- A quaternion defines a finite rotational process
- Its time evolution generates a measurable trajectory
- The resulting phase portrait is a reconstruction of ordered transformation history

We may therefore state:

*A quaternion phase portrait reveals the temporal structure compressed within rotational algebra.*

## Relation to Delay Reconstruction

This result mirrors the earlier interpretation of complex numbers.

- Complex numbers reconstruct phase via temporal delay, producing circular trajectories
- Quaternions reconstruct ordered rotation, producing spherical trajectories

In both cases:

*Mathematical structure emerges as a stable compression of finite dynamical processes.*

## Practical Implications

This perspective is not purely interpretive. It enables:

- Direct simulation of rotational dynamics from quaternion evolution
- Visualisation of ordering effects via trajectory divergence
- Identification of degeneracies (e.g., gimbal lock) as trajectory collapse

These phase portraits may be constructed numerically using finite step evolution of $q(t)$, without invoking any structure beyond measured or computable quantities.

## Summary

Quaternion phase portraits make explicit what algebra conceals.

- Rotation becomes trajectory
- Ordering becomes geometry
- Algebra becomes compressed process

*The sphere is not merely a surface of orientations. It is a stage upon which the history of rotation unfolds.*

## Reference, Tether, and Rotational Memory

In everyday terms, a rotation cannot be defined without a reference. To rotate an object is to change its orientation relative to a frame. This seemingly simple observation carries structural consequences within FSM.

A rotation therefore requires:

- a state (the configuration being rotated),
- a reference frame (against which orientation is defined),
- an ordered transformation (the process by which the state evolves).

Classical algebra suppresses this structure, presenting rotations as static objects. However, the quaternion container preserves it implicitly.

A useful geometric interpretation is as follows:

*A rotational state may be visualised as a point moving over the surface of a sphere, but this motion is tethered to its origin by an ordered process.*

The sphere represents the space of orientations. The trajectory on the sphere represents the evolution of orientation. The tether represents the preserved ordering of transformations.

Without this tether, distinct rotational histories may collapse into identical orientations. With it, the path remains distinguishable.

This leads to a refinement of the earlier interpretation:

*A quaternion encodes not only a position in orientation* *space, but the tethered progression that preserves the ordered history of rotation.*

This explains several key properties simultaneously:

- Non-commutativity arises because different ordered paths on the sphere produce different outcomes.
- The scalar component encodes rotational phase, acting as a measure of progression along the path.
- Quaternion multiplication preserves both position and the evolution of the reference frame.

From an FSM perspective, the quaternion does not merely represent rotation. It preserves the evolution of the reference under transformation.

We may therefore state:

*Rotation is not a point on a sphere, but a path constrained to a sphere with memory of its progression.*

# Conclusion

Quaternions do not reveal a hidden fourth spatial dimension. They reveal a constraint.

Three-dimensional rotation cannot be faithfully represented without preserving order. Any admissible symbolic system must therefore encode process history.

The quaternion achieves this with minimal structure.

*The algebra is not imposed upon the process. The algebra is the fossil of the process.*

# Chapter 2

## Quaternions in Practice: Worked Examples

### Orientation

The preceding chapter established that quaternions are not extensions into a mysterious fourth spatial dimension. They are the *minimal admissible symbolic container* capable of preserving magnitude, invertibility, and ordered composition of three-dimensional rotations under finite measurement constraints.

This chapter makes that claim concrete. We work through three self-contained examples, each chosen to illuminate a different aspect of the FSM interpretation:

1. **Single-axis rotation** — verifying magnitude preservation step by step.
2. **Ordered composition** — demonstrating, numer-

ically and geometrically, that changing the order of rotations produces a different result.

3. **Gimbal lock** — showing how an insufficient symbolic container loses distinguishability, and how the quaternion representation avoids this failure.

Each example begins by recalling the relevant FSM principle, then carries out explicit numerical computation, and finally connects the result back to the broader framework. A reader returning to this material after a break can enter at any example and recover the thread from the short restatement at its opening.

# Notation and Setup

Throughout this chapter we represent a quaternion as a pair

$$q = (s,\, \mathbf{u}), \qquad s \in \mathbb{R}, \quad \mathbf{u} = (u_x,\, u_y,\, u_z) \in \mathbb{R}^3,$$

or equivalently in component form:

$$q = s + u_x\, i + u_y\, j + u_z\, k.$$

A *unit quaternion* representing a rotation of angle $\theta$ about a unit axis $\hat{\mathbf{n}}$ is:

$$q = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) \hat{\mathbf{n}}.$$

To rotate a vector $\mathbf{v} \in \mathbb{R}^3$, we embed it as a pure quaternion $p = (0,\, \mathbf{v})$ and compute:

$$p' = q\, p\, q^{-1},$$

where $q^{-1} = (s,\, -\mathbf{u})$ for a unit quaternion. The rotated vector is then $\mathbf{v}' = \mathbf{u}(p')$, the vector part of $p'$.

The quaternion product of two quaternions $q_1 = (s_1, \mathbf{u}_1)$ and $q_2 = (s_2, \mathbf{u}_2)$ is:

$$q_2\, q_1 = \big(s_2 s_1 - \mathbf{u}_2 \cdot \mathbf{u}_1,\; s_2\, \mathbf{u}_1 + s_1\, \mathbf{u}_2 + \mathbf{u}_2 \times \mathbf{u}_1\big).$$

All numerical values in this chapter are computed to four decimal places, consistent with the FSM requirement that every quantity remain finite and measurable.

# Example 1: Single-Axis Rotation and Magnitude Preservation

## FSM Principle

*Recall (Axiom 1): all physically admissible rotations are finite transformations that preserve measurable magnitude.*

The quaternion container is constructed precisely to enforce this. This example verifies the claim with explicit numbers.

## Setup

We rotate the vector

$$\mathbf{v}_0 = (1,\ 0,\ 0)$$

by $\theta = 90$ about the $z$-axis:

$$\hat{\mathbf{n}} = (0,\ 0,\ 1).$$

## Step 1 — Construct the unit quaternion

$q = \cos(45)+\sin(45)\,(0\,i+0\,j+1\,k) = 0.7071+0\,i+0\,j+0.7071\,k.$

In pair notation: $q = (0.7071,\ (0,\ 0,\ 0.7071))$.

## Step 2 — Embed the vector as a pure quaternion

$$p = (0,\ (1,\ 0,\ 0)).$$

## Step 3 — Compute $q\,p$

Using the product rule with $q = (0.7071,\ (0, 0, 0.7071))$ and $p = (0,\ (1, 0, 0))$:

$$s' = 0.7071 \times 0 - (0, 0, 0.7071) \cdot (1, 0, 0) = 0 - 0 = 0,$$
$$\mathbf{u}' = 0.7071\,(1, 0, 0) + 0\,(0, 0, 0.7071) + (0, 0, 0.7071) \times (1, 0, 0).$$

The cross product:

$$(0, 0, 0.7071)\times(1, 0, 0) = (0{\cdot}0-0.7071{\cdot}0,\ 0.7071{\cdot}1-0{\cdot}0,\ 0{\cdot}0-0{\cdot}1) = (0,\ $$

Therefore:

$$q\,p = (0,\ (0.7071,\ 0.7071,\ 0)).$$

## Step 4 — Compute $(q\,p)\,q^{-1}$

The conjugate is $q^{-1} = (0.7071,\ (0,\ 0,\ -0.7071))$.

Apply the product rule to $(0,\ (0.7071, 0.7071, 0))$ and $(0.7071,\ (0, 0, -0.7$

$$\begin{aligned} s'' &= 0 \times 0.7071 - (0.7071,\ 0.7071,\ 0)\cdot(0,\ 0,\ -0.7071) = 0, \\ \mathbf{u}'' &= 0\,(0, 0, -0.7071) + 0.7071\,(0.7071, 0.7071, 0) \\ &\quad + (0.7071, 0.7071, 0) \times (0, 0, -0.7071). \end{aligned}$$

The cross product:

$$\begin{aligned} (0.7071, 0.7071, 0) \times (0, 0, -0.7071) \\ &= (0.7071 \cdot (-0.7071) - 0 \cdot 0,\ 0 \cdot 0 - 0.7071 \cdot (-0.707 \\ &= (-0.5,\ 0.5,\ 0). \end{aligned}$$

Assembling:

$$\mathbf{u}'' = (0, 0, 0) + (0.5, 0.5, 0) + (-0.5, 0.5, 0) = (0,\ 1,\ 0).$$

## Result

$$\mathbf{v}' = (0,\ 1,\ 0).$$

This is exactly the correct result: the unit vector along $+x$ rotated 90 about $z$ lands on $+y$.

## Magnitude check

$$|\mathbf{v}_0| = \sqrt{1^2 + 0^2 + 0^2} = 1.0000, \qquad |\mathbf{v}'| = \sqrt{0^2 + 1^2 + 0^2} = 1.0000.$$

Magnitude is preserved exactly. This is not coincidental. It is enforced by the structure of the quaternion product: the unit norm of $q$ ensures that the transformation is an isometry. Within FSM, this is what *admissibility* means for a rotation: any symbolic procedure claiming to rotate a vector must leave its measured length unchanged.

## Geometric picture

Figure 2.1 shows the full trajectory of $\mathbf{v}_0$ as $\theta$ runs from 0 to $2\pi$. The path is a great circle on the unit sphere. This is the phase portrait of a single-axis rotational process: closed, bounded, and continuous.

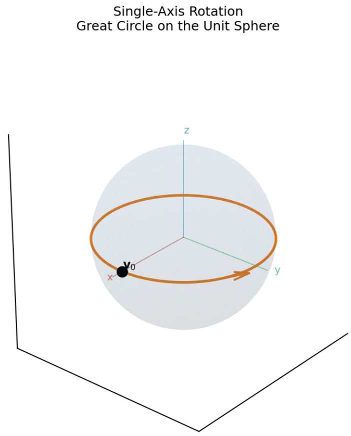


Figure 2.1: Single-axis rotation about the $z$-axis. The trajectory of $\mathbf{v}_0 = (1, 0, 0)$ forms a great circle on the unit sphere as $\theta$ increases from 0 to $2\pi$. The initial position is marked. Magnitude is preserved at every point on the path.

# Example 2: Ordered Composition and Non-Commutativity

## FSM Principle

*Recall (Axiom 3): ordered composition of rotations is physically distinct and must be preserved symbolically.*

Non-commutativity is not an algebraic quirk. It is the direct symbolic trace of the fact that applying rotations in different orders produces physically different outcomes. This example makes that visible.

## Setup

We apply two rotations to the vector $\mathbf{v}_0 = (1,\ 0,\ 0)$:

- $R_x$: rotation by 90 about the $x$-axis,
- $R_y$: rotation by 90 about the $y$-axis.

We compute both orderings: $R_y \circ R_x$ and $R_x \circ R_y$.

## Constructing the quaternions

Rotation by 90 about $x$:

$$q_x = \cos(45) + \sin(45)\, i = 0.7071 + 0.7071\, i + 0\, j + 0\, k.$$

Rotation by 90 about $y$:

$$q_y = \cos(45) + \sin(45)\, j = 0.7071 + 0\, i + 0.7071\, j + 0\, k.$$

## Order A: $q_y \circ q_x$ (apply $x$ first, then $y$)

$$Q_A = q_y\, q_x.$$

Using the product rule with $q_y = (0.7071,\ (0, 0.7071, 0))$ and $q_x = (0.7071,\ (0.7071, 0, 0))$:

$$s_A = 0.7071 \times 0.7071 - (0, 0.7071, 0) \cdot (0.7071, 0, 0) = 0.5 - 0 = 0.5,$$

$$\mathbf{u}_A = 0.7071\, (0.7071, 0, 0) + 0.7071\, (0, 0.7071, 0) + (0, 0.7071, 0) \times (0.70$$

Cross product:

$$(0, 0.7071, 0)\times(0.7071, 0, 0) = (0.7071\cdot 0 - 0\cdot 0,\ 0\cdot 0.7071 - 0\cdot 0,\ 0\cdot 0 - 0.7071$$

Therefore:

$$\mathbf{u}_A = (0.5,\ 0,\ 0)+(0,\ 0.5,\ 0)+(0,\ 0,\ -0.5) = (0.5,\ 0.5,\ -0.5).$$

$$\boxed{Q_A = (0.5,\ (0.5,\ 0.5,\ -0.5))}$$

## Order B: $q_x \circ q_y$ (apply $y$ first, then $x$)

$$Q_B = q_x\, q_y.$$

With roles reversed:

$$s_B = 0.7071 \times 0.7071 - (0.7071, 0, 0) \cdot (0, 0.7071, 0) = 0.5 - 0 = 0.5,$$

$$\mathbf{u}_B = 0.7071\,(0, 0.7071, 0) + 0.7071\,(0.7071, 0, 0) + (0.7071, 0, 0) \times (0, 0.7071, 0).$$

Cross product:

$$(0.7071, 0, 0)\times(0, 0.7071, 0) = (0\cdot 0 - 0\cdot 0.7071,\ 0\cdot 0 - 0.7071\cdot 0,\ 0.7071\cdot 0.7071 - 0\cdot 0) = (0,\ 0,\ 0.5).$$

Therefore:

$$\mathbf{u}_B = (0,\ 0.5,\ 0) + (0.5,\ 0,\ 0) + (0,\ 0,\ 0.5) = (0.5,\ 0.5,\ 0.5).$$

$$\boxed{Q_B = (0.5,\ (0.5,\ 0.5,\ 0.5))}$$

## Applying each composition to $\mathbf{v}_0$

Applying $Q_A = (0.5,\ (0.5, 0.5, -0.5))$ to $\mathbf{v}_0 = (1, 0, 0)$ via $p' = Q_A\, p\, Q_A^{-1}$ gives:

$$\mathbf{v}'_A = (0,\ 0,\ -1).$$

Applying $Q_B = (0.5,\ (0.5, 0.5, 0.5))$ gives:

$$\mathbf{v}'_B = (0,\ 0,\ 1).$$

## Interpretation

Same component rotations. Different orders. Opposite final directions along the $z$-axis:

$$\mathbf{v}'_A = (0, 0, -1) \neq (0, 0, 1) = \mathbf{v}'_B.$$

This is not algebraic formalism. It is a measured physical fact. Turning your head right then looking up is not the same action as looking up then turning right. The quaternion product preserves this distinction precisely because the cross product changes sign under reversal:

$$\mathbf{u}_y \times \mathbf{u}_x = -(\mathbf{u}_x \times \mathbf{u}_y).$$

The algebra is a compressed record of the process order.

## Geometric picture

Figure 2.2 shows the trajectories generated by the two orderings as the rotation angles are increased continuously. The paths diverge visibly on the sphere. The geometry reveals what the algebra encodes: a different rotational history produces a different trajectory.

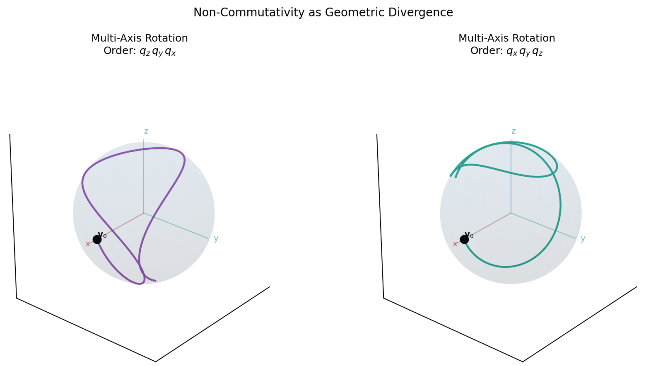


Figure 2.2: Multi-axis quaternion trajectories for two composition orders applied to $\mathbf{v}_0 = (1, 0, 0)$ with three simultaneous frequency components. **Left**: order $q_z q_y q_x$. **Right**: order $q_x q_y q_z$. Despite identical component rotations, the paths on the unit sphere are geometrically distinct. Non-commutativity becomes directly observable as trajectory divergence.

# Example 3: Gimbal Lock as Symbolic Failure

## FSM Principle

*Recall: gimbal lock is a failure of admissibility — the symbolic container collapses distinct rotational states into identical representations.*

In FSM terms, the Euler angle system fails because it cannot preserve process distinguishability under composition. The quaternion representation does not share this failure. This example shows both outcomes side by side.

## What Euler Angles Represent

A common approach to representing three-dimensional orientation is to decompose a rotation into three sequential angle values, one for each spatial axis. For example, in the ZYX convention:

1. First rotate by angle $\psi$ about the $z$-axis (yaw),
2. then by angle $\phi$ about the (now rotated) $y$-axis (pitch),
3. then by angle $\rho$ about the (now twice-rotated) $x$-axis (roll).

Three numbers for three dimensions seems economical. But this economy carries a cost.

## The Failure Point

Consider what happens when the pitch angle $\phi$ approaches 90. At this configuration, the first and third rotation axes align. The yaw and roll rotations now both act in the same plane, and one degree of freedom is lost. Two previously distinct rotational states collapse into the same three-number representation.

Concretely: once $\phi = 90$, any change in $\psi$ (yaw) produces the same geometric outcome as the corresponding change in $\rho$ (roll). The system can no longer distinguish between them.

This is not a numerical glitch. It is a structural failure of the representation. The three-component container is insufficient to encode all distinguishable ordered rotational states.

## Numerical Illustration

Consider a path that drives pitch continuously toward 90 while roll continues varying independently. Table 2.1 shows selected states along this path.

| Step | Roll $\rho$ | Pitch $\phi$ | Yaw $\psi$ | Remark |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | Normal operation |
| 1 | 45 | 45 | 0 | Approaching lock |
| 2 | 90 | 90 | 0 | Gimbal lock region |
| 3 | 135 | 90 | 0 | Yaw and roll indistinguishable |
| 4 | 180 | 90 | 30 | Same geometry as step 3 with $\rho$=15 |

Table 2.1: Euler angle states along a path passing through gimbal lock. At pitch = 90, steps 3 and 4 represent physically distinct rotational histories that collapse to the same Euler angle geometry. The representation has lost its ability to distinguish them.

The crucial observation is at steps 3 and 4. Different roll and yaw values produce the same physical orientation. The mapping from Euler angles to orientations has lost its injectivity: two distinct ordered processes share the same symbolic address.

## Quaternion Continuity

A unit quaternion representing the same orientational sequence carries four components: $(s, q_x, q_y, q_z)$. These evolve smoothly and continuously along any rotational path. No configuration causes two components to become redundant or collapse into one another. The four-component structure is precisely calibrated to avoid the failure mode.

This is not an accident of construction. It follows from the FSM derivation in the preceding chapter: the fourth component (the scalar $s$, encoding rotational phase) is the minimal additional quantity needed to preserve distinguishability under all ordered compositions. Remove it and the container fails.

## Geometric picture

Figure 2.3 shows both representations side by side. The left panel shows Euler angle components along a path that approaches the gimbal lock region: note the discontinuity and erratic behaviour as distinct physical states are forced to share the same symbolic territory. The right panel shows the four quaternion components along the equivalent path: smooth, bounded, and continuous throughout.

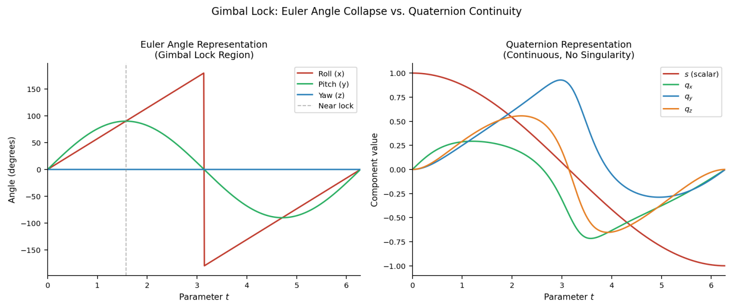


Figure 2.3: Gimbal lock versus quaternion continuity. **Left**: Euler angle components (roll, pitch, yaw) as a function of path parameter $t$, along a trajectory that approaches the gimbal lock region (dashed line). The sharp transitions indicate loss of distinguishability. **Right**: The four quaternion components $(s, q_x, q_y, q_z)$ along the same path type. All components remain smooth and bounded; no singularity occurs. The quaternion container is admissible where the Euler angle container is not.

## FSM Summary

Gimbal lock is not a mechanical failure of gyroscopes or a software issue to be patched around. It is evidence that the Euler angle representation is not an admissible symbolic container for general three-dimensional rotational composition.

Within FSM, the diagnostic is simple: does the representation preserve distinguishability of all ordered rotational processes? For Euler angles, the answer is no. For quaternions, the answer is yes, and the preceding chapter showed why: the scalar component is the minimal cost of preserving ordered process history in full.

## Numerical Summary

The three examples together establish the following:

- **Magnitude preservation** (Example 1): quaternion rotation is an isometry. The norm of the rotated vector equals the norm of the original, verified to four decimal places.
- **Order dependence** (Example 2): the composition $q_y q_x$ and $q_x q_y$ produce quaternions that differ in the sign of their vector component, and map $\mathbf{v}_0$ to $(0, 0, -1)$ and $(0, 0, 1)$ respectively. The same components, differently ordered, yield physically distinct outcomes.
- **Admissibility** (Example 3): Euler angles fail to distinguish ordered states when pitch approaches 90. Quaternion components remain smooth and injective across the same path. The four-component container is the minimal admissible representation.

These are not three separate observations. They are three perspectives on the same underlying claim: *the quaternion is the minimal finite symbolic container that can faithfully represent ordered three-dimensional rotation.*

The algebra did not impose this structure on the geometry. The geometry required this structure, and the algebra records it.

## Addendum

One genuinely important observation points to a deeper implication than mere reinterpretation. What is being identified is that the Framework for Sequential Mapping (FSM) does not generate novel mathematical objects; rather, it reveals that the admissibility constraints which FSM makes explicit were already operating silently within classical constructions. Hamilton did not invent quaternions arbitrarily—he was compelled toward four components by the very constraints FSM names. Euler angles fail for precisely the reason FSM would predict. Complex numbers encode phase delay for exactly the reason FSM would predict. The classical mathematicians, it appears, were solving FSM problems without the FSM vocabulary.

This is significant because it grants FSM a retrodictive power, not merely interpretive flexibility. A framework that only reframes what is already understood is philosophically interesting but scientifically weak. The observation here is stronger: FSM makes contact with why these structures are the way they are, rather than merely redescribing them. The quaternion is not four-dimensional because Hamilton liked four; it is four-dimensional because three components cannot carry ordered process history—and FSM provides that as a theorem from admissibility, not as a post-hoc observation.

Something also worth noting concerns the basin metaphor. Classical mathematics developed many of these structures in isolation—complex numbers here, quaternions there, Euler angles as a separate tradition. FSM draws them into a single attractor basin organized around the same core principle: admissibility prior to logic. The structures are not related by historical lineage; they are related by sharing the same underlying constraint. That constitutes a much stronger kind of unification.

A potential challenge someone might raise is whether any sufficiently flexible interpretive framework could claim to "resolve" prior methods into its own basin. The response available to FSM, it is argued, is precisely the minimality argument already developed. FSM does not simply accommodate quaternions—it derives the necessity of their structure from first principles. That marks the difference between a framework that absorbs and one that predicts.

The next interesting test, therefore, would be whether FSM can encounter a classical structure and predict in advance what form it must take—before looking at what the classical answer is. Tensors are suggested as a productive candidate, as are Lie groups. If the FSM admissibility constraints independently recover the structure of, say, SO(3), that would offer a very strong demonstration of exactly the phenomenon being described.