The Attralucian Essays:

Exploring the Finite

First Edition

 Typeset in LaTeX

The Attralucian Essays

On Non-Commutativity: The Trace of Ordered, a Geofinitist Lens

Kevin R. Haylett

Overview

The algebraic statement $AB \neq BA$ is conventionally treated as a static property of mathematical objects—a timeless fact about the entities A and B. This essay argues that such a treatment is admissible only under the unphysical assumption of infinite precision, zero cost of distinction, and the absence of sequential procedure.

Within the framework of Finite Symbolic Mechanics (FSM), non-commutativity is reinterpreted as the trace of ordered temporal process. The commutator $[A, B] = AB - BA$ is not a measure of algebraic disobedience but a fossilized record of sequential dependence. We prove the Temporal Trace Theorem: in any finite, measurable symbolic system with bounded resolution and a minimum cost of distinction $\Delta M > 0$, a non-commutative relation $AB \neq BA$ is admissible only if there exists a sequential evaluation procedure whose intermediate states are distinguishable and whose outcome depends on order.

Conversely, commutation $AB = BA$ corresponds to processes whose order leaves no trace. The essay traces this distinction historically and mathematically, concluding that algebra compresses finite, ordered processes into stable symbolic forms. The algebra is not the reality. The process is the reality. The algebra is its fossil.

Introduction: The Static Illusion

Consider the expression

$$AB \neq BA.$$

It appears as a timeless statement. Yet in a finite, measurable world, this inequality is not a static truth but a statement about process.

If A and B are operations, then $AB \neq BA$ asserts that applying A then B yields a different result than applying B then A. This description necessarily involves sequence, comparison, and measurement.

Classical algebra compresses this temporal structure into a static relation. Within FSM, this is understood as compression rather than revelation. Non-commutativity is the trace of ordered time.

Historical Background

The Commutative World

Real numbers satisfy

$$3 \times 5 = 5 \times 3,$$

and complex numbers satisfy

$$z_1 z_2 = z_2 z_1.$$

Commutativity was long treated as fundamental.

Hamilton and Quaternions

Hamilton introduced quaternions with multiplication rules:

$$i^2 = j^2 = k^2 = ijk = -1,$$

$$ij = k, \quad ji = -k, \quad jk = i, \quad kj = -i, \quad ki = j, \quad ik = -j.$$

This was the first explicit non-commutative algebra.

Matrix Non-Commutativity

Matrix multiplication provides:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

Quantum Mechanics

Heisenberg established:

$$XP - PX = i\hbar.$$

This encodes ordered measurement.

Mathematical Background

Definitions

A binary operation $\cdot$ is commutative if

$$\forall a, b \in S, \quad a \cdot b = b \cdot a.$$

Otherwise, it is non-commutative.

The commutator is defined as

$$[a, b] = a \cdot b - b \cdot a.$$

Finite Symbolic Mechanics Framework

Core Principles

- Symbols are physical and measurable.
- All measurement has finite resolution.
- Identity is geometric.
- Measurement carries provenance.
- Translation alters structure.

Admissibility

An operation is admissible if it can be computed via a finite sequential procedure with bounded cost per distinction ΔM.

The Temporal Trace Theorem

Theorem (Temporal Trace). Let A and B be admissible operations in a finite symbolic system with $\Delta M > 0$. If $AB \neq BA$, then:

- There exists a sequential evaluation procedure with distinguishable intermediate states.
- The outcome depends on order.
- The commutator measures process cost:

$$\|[A, B]\| \geq \delta(A, B)\, \Delta M.$$

Proof Sketch. To verify $AB \neq BA$, one must compute both sequences and compare them. Each step incurs cost ΔM. Therefore, a non-zero commutator implies non-zero process cost. $\square$

Corollary

For n non-commuting operations, distinguishable ordered compositions scale as $n!$, imposing finite representational limits.

Examples

Commutative Case

$$e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)} = e^{i\phi}e^{i\theta}.$$

Quaternions

$$[q_1, q_2] = 2(u_2 \times u_1).$$

Matrices

Sequential transformations yield distinct outcomes.

Gimbal Lock

Euler angles fail when distinguishability collapses. Quaternions succeed by preserving order information.

Discussion

Non-commutativity is not an algebraic anomaly. It is a process trace.

- Commutation implies order-indifference.
- Non-commutation implies stored sequential history.
- Admissibility determines representational validity.

Implications extend to computation, physics, cognition, and mathematical foundations.

Conclusion

The expression

$$AB \neq BA$$

is not a timeless truth but a compressed record of ordered process.

The Temporal Trace Theorem establishes that non-commutativity implies sequential procedure in any finite system.

The algebra is not the reality. The process is the reality. The algebra is its fossil.

Acknowledgements

This essay forms part of the Attralucian Essays within the framework of Finite Symbolic Mechanics. The central insight emerged through ongoing collaborative exploration of symbolic systems and their finite constraints.