The Attralucian Essays:

Exploring the Finite

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Zeno's Paradoxes: A Geofinitist Reinterpretation

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Overview

Zeno's paradoxes challenge the intelligibility of motion by invoking infinite subdivision, instantaneous rest, and relative displacement. Classical mathematics addresses these puzzles through convergence, limits, and calculus. Geofinitism does not reject these tools; rather, it reframes the paradoxes as arising when ideal limiting procedures are treated as physical requirements.

Within a Geofinitist framework, motion is a finite, measured trajectory with spatial and temporal resolution. Infinite subdivision becomes a useful fiction, not a demand placed upon physical motion.

The Classical Puzzles

In *Achilles and the Tortoise*, Achilles must first reach the tortoise's previous position, by which time the tortoise has moved forward. Repeating this argument appears to generate infinitely many tasks before overtaking can occur.

In *the Dichotomy*, motion toward a target seems to require first traversing half the distance, then half the remainder, and so on without end.

In *the Arrow*, an arrow at any instant occupies a definite position and appears motionless, raising the question of how motion arises from instants of rest.

In *the Stadium*, relative motion seems to produce conflicting accounts of displacement depending on frame and comparison.

Classically, these are addressed by convergent series and the calculus of limits. Geofinitism accepts these as powerful mathematical tools, while insisting that physical motion is observed only through finite measurement.

Source of the Paradox

Zeno's arguments depend on several idealisations:

- infinite subdivision of distance and time,
- pointwise instants treated as physically complete states,
- exact equality as the criterion for arrival,
- absence of measurement resolution,
- confusion between mathematical limiting descriptions and physical procedures.

The paradoxes arise when a limiting abstraction is mistaken for a required sequence of physical actions.

Geofinitist Principles Applied

1. Motion as Trajectory

Motion is not a sequence of isolated point-visits. It is a trajectory through a finite kinematic state space, observed through intervals and changes.

2. Measured Position and Time

Positions and times are measured with finite resolution:

$$x(t) \pm \varepsilon_x, \qquad t \pm \varepsilon_t.$$

3. Layered Dynamics

Motion occurs across scales: muscular control, gait, sensor sampling, macroscopic displacement. These layers form a finite cascade rather than an infinite regress.

4. Infinity as Useful Fiction

Infinite subdivision is a modelling tool. It is valid where it yields stable predictions, but it is not itself a physical operation.

5. Finite Resolution

All observations occur above resolution floors:

$$\Delta x_{\min} > 0, \qquad \Delta t_{\min} > 0.$$

Measured Kinematics

Let x_t be a measured position sampled at discrete time intervals:

$$x_t = (v_t, \varepsilon_{x,t}, P_{x,t}) \in \mathbb{M}, \qquad t = 0, 1, \ldots, T.$$

Measured velocity is defined over a finite interval:

$$\dot{x}_t = \left(\frac{v_{t+1} - v_t}{\Delta t_{\min}}, \frac{\varepsilon_{x,t} + \varepsilon_{x,t+1}}{\Delta t_{\min}}, P_{\dot{x}} \right).$$

Motion is detected when the velocity band excludes zero.

The Dichotomy

Let $L > 0$ be a target displacement. Define cumulative path length:

$$S_n = \sum_{k=0}^{n-1} |v_{k+1} - v_k|.$$

The remaining distance is:

$$R_n = L - S_n.$$

The process terminates when:

$$R_n \leq \Delta x_{\min} + \bar{\varepsilon}_x.$$

Thus the required number of measured steps is finite:

$$n^\star = \inf\{n : R_n \le \Delta x_{\min} + \bar{\varepsilon}_x\}.$$

Achilles and the Tortoise

Let $x_t^{(A)}$ and $x_t^{(T)}$ be measured positions of Achilles and the tortoise. Define the measured gap:

$$g_t = x_t^{(T)} - x_t^{(A)}.$$

Its update is:

$$g_{t+1} = g_t + v_t^{(T)} \Delta t_{\min} - v_t^{(A)} \Delta t_{\min} \pm \varepsilon_{g,t}.$$

The catch-up stopping time is:

$$\tau^\star = \inf\{t : g_t \le \Delta x_{\min} + \bar{\varepsilon}_g\}.$$

If Achilles maintains a speed advantage:

$$v_t^{(A)} - v_t^{(T)} \ge \nu > 0,$$

then:

$$\tau^\star \le \left\lceil \frac{g_0}{\nu \Delta t_{\min}} \right\rceil < \infty.$$

Overtaking is therefore a finite stopping event, defined within measurement tolerance.

The Arrow

The arrow paradox assumes that motion must be defined at a durationless instant. Geofinitism replaces this with finite-window velocity:

$$\dot{x}_t = \frac{x_{t+1} - x_t}{\Delta t_{\min}} \pm \varepsilon_{\dot{x}}.$$

The arrow moves when:

$$0 \notin \dot{x}_t \pm \varepsilon_{\dot{x}}.$$

Thus motion is not inferred from frozen instants, but from measured change across finite intervals.

The Stadium

The Stadium paradox concerns relative displacement. In Geofinitism, relative motion is evaluated through measured frame transformations:

$$x_t^{(A|B)} = x_t^{(A)} - x_t^{(B)} \pm \varepsilon_{\text{rel}}.$$

Apparent contradictions are resolved by declaring the measurement frame, sampling interval, and transformation error. Relative motion is not frame-free; it is a measured relation between trajectories.

Interpretation

Under this framework:

- arrival is not exact equality but contact within tolerance,
- velocity is not pointwise metaphysics but finite-window measurement,
- infinite subdivision is a limit process, not a physical task,
- motion is a trajectory, not a sequence of static instants.

Conclusion

Zeno's paradoxes do not demonstrate that motion is impossible. They expose the danger of treating infinite symbolic subdivision as a physical requirement.

Classical calculus resolves the mathematical structure through convergence. Geofinitism complements this by grounding motion in finite measurement: bodies move when trajectories change across resolvable intervals, and events occur when thresholds are crossed within tolerance.

Context. Zeno's paradoxes are treated as boundary confusions between ideal limiting constructions and finite measured motion.

Resolution Scales.

$$\Delta x_{\min} > 0, \qquad \Delta t_{\min} > 0.$$

Measured Position.

$$x_t = (v_t, \varepsilon_{x,t}, P_{x,t}) \in \mathbb{M}.$$

Measured Velocity.

$$\dot{x}_t = \left(\frac{v_{t+1} - v_t}{\Delta t_{\min}}, \frac{\varepsilon_{x,t} + \varepsilon_{x,t+1}}{\Delta t_{\min}}, P_{\dot{x}} \right).$$

Finite Arrival Criterion. For target displacement L:

$$R_n = L - S_n, \qquad R_n \le \Delta x_{\min} + \bar{\varepsilon}_x.$$

Catch-Up Criterion. For Achilles and the tortoise:

$$\tau^\star = \inf\{t : g_t \le \Delta x_{\min} + \bar{\varepsilon}_g\}.$$

Arrow Criterion. Motion holds when:

$$0 \notin \dot{x}_t \pm \varepsilon_{\dot{x}}.$$

Collapse Note. As $\Delta x_{\min}, \Delta t_{\min}, \varepsilon \to 0$, the finite protocol approaches the classical limit of convergent series and derivatives. **Interpretation.** Zeno's paradoxes arise from mistaking infinite mathematical refinement for finite physical procedure. In $\mathbb{M}$, motion is a measured trajectory with stopping criteria.