The Attralucian Essays:

Exploring the Finite

First Edition

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The Banach-Tarski Paradox: A Geofinitist Reinterpretation

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Overview

The Banach–Tarski paradox demonstrates that a solid ball in $\mathbb{R}^3$ can be decomposed into finitely many disjoint pieces and reassembled into two copies of the original using rigid motions. While mathematically consistent within set-theoretic frameworks that include the Axiom of Choice, the result conflicts with physical intuition about volume and conservation.

This paper does not dispute the classical theorem. Instead, it reinterprets it through a Geofinitist lens: bodies, decompositions, and transformations are treated as finite, measurable processes with bounded precision and explicit provenance. Under these constraints, the Banach–Tarski construction is not contradictory but *inadmissible*.

Classical Formulation

Banach and Tarski (1924) showed that a ball $B \subset \mathbb{R}^3$ can be partitioned into finitely many sets $\{S_i\}_{i=1}^k$ such that, under rigid motions T_i,

$$\bigcup_{i=1}^{k} T_i(S_i) = B_1 \cup B_2,$$

where B_1 and B_2 are disjoint copies of B.

The construction relies on:

- the Axiom of Choice,
- non-measurable sets,
- the non-amenability of SO(3).

It does not violate measure theory, but shows that Lebesgue measure cannot extend to all such sets.

Source of the Paradox

The paradox depends on the following assumptions:

- Arbitrary (non-measurable) set construction,
- Exact rigid motions on point sets,
- Infinite precision in decomposition,
- Absence of scale or resolution constraints.

These assumptions enable constructions that cannot be realized or verified within finite procedures.

Geofinitist Principles Applied

1. Finite Representation

Bodies are represented at finite resolution. A physical or computational object must admit a finite description.

2. Measured Decomposition

All partitions must consist of measurable components with nonzero volume:

$$\mathrm{vol}(S_i) \geq \delta V > 0.$$

3. Scale Dependence

Geometry is constructed across levels (voxels $\rightarrow$ regions $\rightarrow$ bodies), with consistency enforced across scales.

4. Operational Identity

Object identity is defined through measurable invariants (e.g. volume, topology) within tolerance bounds.

5. Finite Realizability

Transformations must be implementable with bounded precision and error.

Geofinitist Reformulation

Let B be a ball of volume $V(B)$. A valid decomposition satisfies:

- finite partition $\{S_i\}_{i=1}^{k}$,
- $\mathrm{vol}(S_i) \geq \delta V$,
- $k \leq N = V(B)/\delta V$,

- measurable transformations T_i with bounded error.

Define the reconstructed body:

$$U = \bigcup_{i=1}^{k} T_i(S_i).$$

Volume is preserved up to uncertainty:

$$\mathrm{vol}(U) \approx V(B) \pm \sigma.$$

Where the Paradox Breaks

Under these constraints:

- non-measurable sets are excluded,
- infinite precision decompositions are disallowed,
- transformations carry bounded error,
- total volume is conserved within tolerance.

Thus the Banach–Tarski duplication is not realized. The construction requires objects and operations outside the admissible domain.

Interpretation

In this framework:

- Bodies are finite, measurable entities,
- Decompositions are constrained by resolution,

- Transformations are approximate, not exact,
- Volume is a measured invariant.

The paradox becomes a statement about the limits of symbolic freedom rather than a property of physical geometry.

Conclusion

The Banach–Tarski theorem remains a valid result within classical set theory. Geofinitism reframes its significance: it identifies constructions that exceed finite, measurable admissibility.

Under finite constraints, volume is conserved and paradoxical duplication is excluded. The result is not a contradiction, but a boundary condition on symbolic decomposition.

Context. Banach–Tarski relies on non-measurable sets and idealized rigid motions. In $\mathbb{M}$, bodies are finite, measurable, and decompositions carry provenance.

Measured Body. At resolution $\eta > 0$, represent S by voxels:

$$S_\eta = \bigcup_{j=1}^{N(\eta)} V_j.$$

Measured Volume.

$$\mu_{\mathbb{M}}(A) = \left(\sum_{V_j \subseteq A} \eta^3, \; \varepsilon_\mu(A) \right).$$

Finite Additivity.

$$\mu_{\mathbb{M}}(A \cup B) \approx \mu_{\mathbb{M}}(A) + \mu_{\mathbb{M}}(B).$$

Admissible Motions. Rigid motions are implemented with tolerance:

$$\mu_{\mathbb{M}}(TA) \approx \mu_{\mathbb{M}}(A).$$

Conservation Law.

$$\mu_{\mathbb{M}}(U) \approx \mu_{\mathbb{M}}(S_\eta).$$

Result. No admissible decomposition yields:

$$\mu_{\mathbb{M}}(U) \approx 2\, \mu_{\mathbb{M}}(S_\eta).$$

Collapse Note. Removing measurability and finite constraints restores the classical paradox.

Interpretation. Banach–Tarski is a non-