The Principia Geometrica

Finite Symbolic Mechanics II

Measured Structures

Compressed Operations, Measured Number Classes, Alphonic Sets, and Dimensional Stabilisation

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Author's Note on Status

This document is a working seed for a second volume of *The Principia Geometrica: Finite Symbolic Mechanics*. It is not intended as a completed formal system. It gathers a set of emerging ideas that appear to follow naturally from the first volume: Measured Numbers, the Alphonic Limit, the Generon, the Spherical Uncertainty Distribution, the abacus as arithmetic archetype, and the Finite-Symbol Embedding Theorem.

The central purpose of this draft is to make explicit a missing hinge. In the first volume, a number is already treated as a Measured Number

$$m = (v, \varepsilon, P),$$

where v is nominal value, $\varepsilon > 0$ is uncertainty, and P is provenance. The present volume asks what follows once operations, number classes, sets, functions, and dimensional structures are also treated as finite measured-symbolic trajectories.

The guiding observation is simple:

> A mathematical expression is a compressed record of an unfolded symbolic trajectory.

Thus a^2, $a \times b$, p/q, $\sqrt{x}$, π, e, set membership, functional mapping, and dimensional description are not first treated as completed objects. They are treated as finite symbolic processes whose realised outputs terminate under Alphonic constraint.

Where definitions appear provisional, they should be read as stabilising markers rather than final closures. The task of this document is to open the next basin of work: a finite theory of measured structures.

Simul Pariter

Contents

Preface: From Measured Numbers to Measured Structures

The first volume of Finite Symbolic Mechanics develops the measurement-first foundation. It argues that formal and communicable mathematics begins only after a finite symbol has been admitted. It introduces the Generon as the process by which pre-symbolic potential becomes admissible symbol, the Alphon as the finite alphabet or representational base, the Alphonic Limit as the minimum distinguishable symbolic region, and the Measured Number as the primary numerical object.

This second volume begins from a further observation. If the number is measured, then the operation is also measured. If the operation is measured, the set is measured. If the set is measured, then membership, function, relation, dimension, convergence, and physical law must also carry uncertainty, provenance, and admissibility conditions.

This is not a rejection of classical mathematics. It is a relocation of classical mathematics into the limiting case of finite measured-symbolic process. Classical mathematics remains useful because its compressed expressions are extraordinarily powerful. Yet the compression can become misleading when the hidden trajectory is forgotten.

A child learns that

$$a^2 + b^2 = c^2.$$

The expression becomes familiar and appears almost static. But each symbol carries a buried process. Squaring hides multiplication. Multiplication hides addition, counting, geometric containment, or substrate-specific procedure. The square root hides an inverse search trajectory. Equality hides tolerance-bound containment equivalence. The real number hides an ideal limit that is never realised as a finite measurement.

The present volume unfolds these compressions. It asks what mathematics looks like when the trajectory is restored.

Relationship to Existing FSM and FSET Documents

This document is designed to sit after the first FSM monograph and alongside the Finite-Symbol Embedding Theorem (FSET) paper.

The first FSM monograph establishes the foundational objects:

$$m = (v, \varepsilon, P), \qquad \varepsilon > 0,$$

finite symbols, approximate equality, provenance composition, the Alphon, the Alphonic Limit, the Spherical Uncertainty Distribution, and the metrological tether between exogenous measurement and endogenous symbol.

The FSET paper establishes that finite symbolic systems may be reconstructed geometrically from finite observations by replacing the classical smooth-manifold assumptions of Takens' theorem with finite representability, bounded uncertainty, and geometric stability under embedding dimension.

This second volume connects those two directions. It proposes that arithmetic operations themselves are finite symbolic dynamical systems. A square-root algorithm, a decimal expansion of π, a continued fraction, a proof trajectory, or a set-construction rule is not merely an expression. It is an unfolding in symbolic phase space. Under a finite Alphon, bounded container, and specified resolution, that unfolding terminates as an admissible measured-symbolic object.

Part I

The Missing Hinge

Chapter 1

Compressed Operations and Symbolic Trajectories

> The sign is short because the path is long.
>
> — Finite Symbolic Mechanics

1.1 The Problem

Classical notation gives the appearance of stillness. An expression such as

$$a^2 + b^2 = c^2$$

appears as a static relation among completed quantities. Within Finite Symbolic Mechanics, this appearance is misleading. The expression is not primary. It is a compressed symbolic residue of a trajectory of operations.

The term a^2 is already a compression. It abbreviates a rule-governed unfolding:

$$a^2 \rightsquigarrow a \times a.$$

But multiplication is also compressed. In natural-number counting it may unfold as repeated addition:

$$a \times a \rightsquigarrow \underbrace{a + a + \cdots + a}_{a \text{ times}}.$$

In measured geometry, however, multiplication is not merely repeated addition. It is a geometric construction. The expression

$$a \times b$$

denotes the construction of a rectangular containment relation from two measured linear symbols. The result is not simply a scalar. It is a new measured symbol with value, uncertainty, and provenance.

The square-root operation is an even deeper compression. The expression

$$c = \sqrt{x}$$

is not merely a value statement. It invokes a rule-governed inverse trajectory: find, construct, or compute an admissible measured symbol whose square is Alphoncially equivalent to x.

1.2 Compressed and Unfolded Forms

Definition 1.1 (Compressed Symbolic Operation). *A compressed symbolic operation is a finite notation that stands for a rule-governed sequence of symbolic transformations.*

Definition 1.2 (Unfolded Symbolic Trajectory). *Let Ω be a symbolic operation and let $m_0 \in \mathcal{M}$ be an admissible measured-symbolic input. The unfolded symbolic trajectory of Ω from m_0 is a sequence*

$$\mathcal{T}_\Omega(m_0) = (m_0, m_1, m_2, \ldots, m_N),$$

where each $m_i \in \mathcal{M}$ is produced by a finite rule and N is the termination stage determined by the relevant Alphonic constraint.

The operation therefore has two forms:

$$\text{compressed form:} \qquad \Omega(m),$$

$$\text{unfolded form:} \qquad m_0 \to m_1 \to m_2 \to \cdots \to m_N.$$

Classical mathematics normally manipulates the compressed form. FSM requires that the unfolded form remain available in principle, because only the unfolded form carries the finite process by which the symbol becomes admissible.

1.3 The Pythagorean Relation Revisited

The familiar expression

$$a^2 + b^2 = c^2$$

compresses several layers of symbolic activity:

$$a, b \longrightarrow a \times a,\ b \times b \longrightarrow a^2,\ b^2 \longrightarrow a^2 + b^2 \longrightarrow c^2 \longrightarrow c.$$

The final transition

$$c^2 \longrightarrow c$$

is not trivial. It invokes a root-generating procedure. If $a = b = 1$, the classical relation gives

$$c = \sqrt{2}.$$

But in FSM this is not read as the recovery of a completed real object. It is read as the unfolding of a square-root generon whose realised output terminates at Alphonic resolution.

1.4 Compression and Forgetting

A central danger of mature notation is that it hides its own origin. Once a symbolic trajectory becomes familiar, it is compressed into a sign. The sign is then mistaken for a primitive object.

Thus:

$$a^2$$

is mistaken for a simple value,

$$\sqrt{x}$$

is mistaken for a completed number,

$$=$$

is mistaken for exact identity, and

$$\mathbb{R}$$

is mistaken for a measurable continuum.

FSM reverses this forgetting. It treats every compressed symbol as a recoverable trajectory through measured-symbolic space. The task is not always to unfold the trajectory in practice. The task is to remember that the trajectory is present.

Chapter 2

Alphonic Termination and the Abacus Principle

2.1 The Abacus as Archetype

The abacus is not merely a historical calculating tool. It is the archetype of finite arithmetic. A bead configuration is a finite symbolic state. A calculation is a physical trajectory through such states. When the beads stop, the calculation has not approximated a hidden Platonic result. It has produced the admissible result available to that apparatus.

This principle generalises. A pencil-and-paper computation, a digital register, a decimal expansion, a neural representation, and an algebraic proof are all finite symbolic processes. They have finite substrates, finite resolution, finite memory, and finite admissibility conditions.

The abacus therefore gives the key rule:

Principle 2.1 (The Abacus Principle)**.** *Every realised arithmetic operation is a finite trajectory through a bounded symbolic state space. Its output is the terminal admissible configuration of that trajectory, not a completed infinite object.*

2.2 Alphonic Termination

Let α denote the Alphonic resolution of a given symbolic or computational substrate. A trajectory

$$(m_0, m_1, m_2, \ldots)$$

terminates Alphoncially when further unfolding no longer produces a distinguishable measured-symbolic state.

Definition 2.1 (Alphonic Termination)**.** *Let $\mathcal{G}$ be a generative rule producing a sequence of Measured Numbers*

$$m_0, m_1, m_2, \ldots \in \mathcal{M}.$$

The Alphonic termination of $\mathcal{G}$ at resolution α, denoted

$$\mathrm{Term}_\alpha(\mathcal{G}),$$

is the first admissible state m_N such that

$$m_{N+1} \approx_\alpha m_N.$$

The output m_N is the finite measured-symbolic value of the generon at resolution α.

If $m_i = (v_i, \varepsilon_i, P_i)$, then a practical termination condition may be written as

$$|v_{N+1} - v_N| < \varepsilon_N + \varepsilon_{N+1} + \delta_\alpha,$$

where δ_α is the Alphonic distinction threshold of the substrate.

This is not a numerical inconvenience. It is a foundational condition. Every physical or computational process terminates at finite representational resolution. Even when a rule can be written so that it continues indefinitely in the classical formal system, its realised symbolic trajectory reaches a point at which no further distinction is admissible within the substrate.

2.3 Root Extraction as an Abacus Process

A square-root algorithm running on a digital processor is an abacus in another form. It moves through finite representational states until further motion falls below the distinction available to that substrate.

For example, the Babylonian square-root rule for $x = 2$ may be written

$$r_{n+1} = \frac{1}{2}\left(r_n + \frac{2}{r_n}\right).$$

Classically, the sequence is said to converge to $\sqrt{2}$. FSM reads the sequence differently. It is a symbolic trajectory in measured-number space. It does not produce a completed infinite object. It produces an admissible measured output at Alphonic termination:

$$\sqrt{2}_\alpha = \mathrm{Term}_\alpha(\mathcal{G}_{\sqrt{\ }}(2)).$$

Thus

$$\sqrt{2}_\alpha = (v_\alpha, \varepsilon_\alpha, P_{\sqrt{2}}),$$

where v_α is the terminal nominal value, ε_α is the uncertainty at the Alphonic Limit, and $P_{\sqrt{2}}$ records the provenance of the generative rule, substrate, precision, and stopping condition.

The FSM statement is therefore not

$$(\sqrt{2})^2 = 2$$

as an exact identity between completed objects. Rather, it is

$$(\sqrt{2}_\alpha)^2 \approx_\alpha 2_\alpha.$$

2.4 Termination Is Not Approximation

The phrase "approximation" can mislead. In the classical frame, a finite decimal is treated as an approximation to a completed real number. In FSM, the finite measured output is not secondary. It is the realised object.

The classical object names an ideal direction of a generon. The measured object is the terminal finite state of that generon in a specified container at a specified Alphon.

This distinction is central:

$$\sqrt{2} \quad \text{names the classical ideal direction,}$$

$$\mathcal{G}_{\sqrt{2}} \quad \text{names the generating rule,}$$

$$\sqrt{2}_\alpha = (v_\alpha, \varepsilon_\alpha, P_{\sqrt{2}}) \quad \text{names the realised Measured Irrational.}$$

Part II

Measured Number Classes

Chapter 3

The Finite Measured-Number Container

3.1 The Parent Space

The parent object remains the Space of Measured Numbers:

$$\mathcal{M} = \{m = (v, \varepsilon, P) : v \in \mathbb{Q},\ \varepsilon \in \mathbb{Q}^+,\ P \in \mathcal{P}\}.$$

Each element $m \in \mathcal{M}$ is a Measure: a finite symbolic object carrying nominal value, uncertainty, and provenance.

It is important not to define $\mathcal{M}$ merely as the union of measured integers, measured rationals, and measured irrationals. That would allow the old classical hierarchy to determine the new framework. Instead, $\mathcal{M}$ is the parent measured-number space. Number classes are provenance classes within $\mathcal{M}$.

3.2 Bounded Containers

A realised symbolic system does not have access to all of $\mathcal{M}$. It has access only to a bounded container.

Definition 3.1 (Representational Container)**.** *A representational container B is a finite physical, computational, or symbolic substrate with bounded storage, finite alphabet, finite precision, and specified admissibility rules.*

Examples include an abacus, a sheet of paper, a fixed-width digital register, a decimal notation with prescribed length, a measuring apparatus, a human working-memory context, or a formal document.

Definition 3.2 (Finite Measured-Number Container)**.** *Given an Alphon α and bounded representational container B, the finite measured-number container is*

$$\mathcal{M}_{\alpha,B} = \{m \in \mathcal{M} : m \text{ is admissibly representable in } B \text{ at resolution } \alpha\}.$$

Proposition 3.1 (Finite Container Proposition)**.** *For fixed α and bounded B, the set $\mathcal{M}_{\alpha,B}$ contains only finitely many distinguishable measured-symbolic states.*

Proof sketch. A bounded container has finite representational capacity. The Alphon specifies a finite alphabet or finite basis of symbolic distinction. At resolution α, symbols separated by less than the Alphonic distinction threshold are not distinguishable. Therefore only finitely many distinguishable configurations can be admitted. Each admissible Measured Number corresponds to at least one such configuration. Hence $\mathcal{M}_{\alpha,B}$ is finite. □

3.3 The Classical Limit

The classical continuum is recovered as an ideal direction:

$$\mathcal{M}_{\alpha,B} \longrightarrow \mathbb{R}$$

only by allowing the uncertainty to tend toward zero, the container to become unbounded, provenance to be discarded, and the distinction threshold to vanish. FSM regards this as a useful fiction, not as a physically realised state.

Chapter 4

Measured Naturals, Integers, and Rationals

4.1 Provenance Rather Than Platonic Membership

Classical number systems classify numbers by ideal membership:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}.$$

FSM classifies measured numbers by finite generative provenance.

Let $\mathfrak{G}_X$ denote a class of generons associated with a classical number type X. Then define

$$\mathcal{M}_X = \{m \in \mathcal{M} : \mathrm{Prov}(m) \in \mathfrak{G}_X\},$$

where X may be N, Z, Q, I, T, or another specified provenance class.

At fixed Alphon and bounded container:

$$\mathcal{M}_{X,\alpha,B} = \mathcal{M}_X \cap \mathcal{M}_{\alpha,B}.$$

Since $\mathcal{M}_{\alpha,B}$ is finite, each $\mathcal{M}_{X,\alpha,B}$ is finite.

4.2 Measured Naturals

Definition 4.1 (Measured Natural). *A Measured Natural is a Measured Number produced by a counting generon. It has the form*

$$n_M = (v, \varepsilon, P_N),$$

where P_N records a finite counting, tallying, or successor-based provenance.

A Measured Natural is not the completed abstract natural number n. It is the finite symbolic state produced by a counting process under Alphonic constraint.

At sufficient resolution, a Measured Natural may be sharply distinguishable from neighbouring measured naturals. But the distinction is still finite, not infinite. Its identity is tolerance-bound.

4.3 Measured Integers

Definition 4.2 (Measured Integer). *A Measured Integer is a Measured Number produced by a signed counting generon. It has the form*

$$z_M = (v, \varepsilon, P_Z),$$

where P_Z records the construction of a direction or sign relative to an admitted zero region.

The classical integer system arises by extending the natural-number basin to include additive inverses. FSM adds that the sign itself is a finite symbolic distinction. It is not free of representation. A negative integer is not merely a point left of zero on an ideal line. It is a measured-symbolic object produced by a signed generon in a container with a zero region and an orientation convention.

4.4 Measured Rationals

Definition 4.3 (Measured Rational). *A Measured Rational is a Measured Number produced by a ratio-generating or division-generating procedure whose provenance records a finite relation between measured integer-like states.*

Thus

$$q_M = (v, \varepsilon, P_Q)$$

may be associated with a classical expression p/q, but the FSM object is not the ideal ratio alone. It includes the finite process by which numerator, denominator, division procedure, precision, and admissibility were established.

Division is especially important because it exposes the boundary of symbolic admissibility. A denominator is not merely nonzero in the classical sense. It must be outside the Alphonic zero region of the relevant container.

Principle 4.1 (Measured Division Principle). *A division operation is admissible only when the denominator is distinguishable from the zero region at the relevant Alphonic resolution.*

4.5 Overlap Between Classes

The same nominal value may be generated by different provenance classes. For example,

$$(1.5, \varepsilon, P_Q)$$

and

$$(1.5, \varepsilon, P_{\text{instrument}})$$

may share nominal value and uncertainty while differing in provenance. They are not automatically identical FSM objects.

This gives the general rule:

Principle 4.2 (Provenance Classification Principle). *Measured-number classes are determined by generative provenance, not by nominal value alone.*

Chapter 5

Measured Irrationals

5.1 The Classical Rupture

The diagonal of the unit square has long served as the historical entry point into irrationality. Classically, if the square has side length 1, the diagonal satisfies

$$d^2 = 1^2 + 1^2 = 2,$$

so

$$d = \sqrt{2}.$$

The classical proof that $\sqrt{2} \notin \mathbb{Q}$ shows that no ratio of integers closes the diagonal exactly.

FSM restates the lesson. The diagonal did not reveal a defect in geometry. It revealed the failure of a symbolic system to close all geometric measurement trajectories inside rational form.

5.2 Irrational Generons

Definition 5.1 (Irrational Generon)**.** *An Irrational Generon is a finite symbolic rule whose classical interpretation generates a non-terminating, non-periodic expansion not expressible as rational closure within the inherited symbolic system.*

Examples include square-root generons for nonsquare integers, certain continued-fraction rules, diagonal constructions, and other finite procedures whose classical ideal directions are irrational.

5.3 Measured Irrationals

Definition 5.2 (Measured Irrational)**.** *A Measured Irrational is the Alphonic termination of an Irrational Generon. It is an element of $\mathcal{M}$, not an element of a completed real continuum:*

$$\mathcal{M}_I = \{\mathrm{Term}_\alpha(\mathcal{G}) : \mathcal{G} \in \mathfrak{G}_I\}.$$

At fixed α and bounded B,

$$\mathcal{M}_{I,\alpha,B} = \mathcal{M}_I \cap \mathcal{M}_{\alpha,B}.$$

For example,

$$\sqrt{2}_\alpha \in \mathcal{M}_{I,\alpha,B}$$

when it is produced by a square-root generon within a specified container.

This is not an approximation to a measurable completed real number. It is the measured number produced by the finite unfolding of the square-root generon at Alphonic resolution.

5.4 The Unit Diagonal as Measured Irrational

Let

$$1_\alpha = (1, \varepsilon_1, P_1)$$

be an admitted measured unit. The diagonal-generating trajectory is

$$1_\alpha, 1_\alpha \longrightarrow 1_\alpha^2 + 1_\alpha^2 \longrightarrow 2_\alpha \longrightarrow \sqrt{2}_\alpha.$$

The diagonal is therefore not an inaccessible completed object. It is an admissible Measured Irrational:

$$d_\alpha = \sqrt{2}_\alpha.$$

The Pythagorean crisis can now be restated:

> The diagonal of the unit square forced mathematics to distinguish between a compressed ideal symbol and an unfolded measured-symbolic trajectory.

5.5 Measured Irrationals Are Finite at Fixed Alphon

Corollary 5.1 (Finite Measured Irrationals)**.** *For fixed α and bounded B, the set $\mathcal{M}_{I,\alpha,B}$ is finite.*

Proof. Since $\mathcal{M}_{I,\alpha,B} \subseteq \mathcal{M}_{\alpha,B}$, and $\mathcal{M}_{\alpha,B}$ is finite, $\mathcal{M}_{I,\alpha,B}$ is finite. □

This is not a claim that there are finitely many classical irrational directions. It is the claim that a finite symbolic container admits only finitely many realised Measured Irrationals.

Chapter 6

Measured Transcendentals and Constants

6.1 Beyond Irrational Generons

Classically, transcendental numbers are numbers that are not roots of any nonzero polynomial equation with integer coefficients. In FSM, this definition belongs to the classical ideal hierarchy. A measured counterpart must be provenance-based.

Definition 6.1 (Transcendental Generon)**.** *A Transcendental Generon is a finite symbolic rule whose classical ideal direction is associated with a transcendental constant, function, or expansion.*

Examples may include generons for π, e, trigonometric constants, exponential-function evaluations, and other classical transcendental objects.

Definition 6.2 (Measured Transcendental)**.** *A Measured Transcendental is the Alphonic termination of a Transcendental Generon:*

$$\mathcal{M}_T = \{\mathrm{Term}_\alpha(\mathcal{G}) : \mathcal{G} \in \mathfrak{G}_T\}.$$

Thus π_α is not a completed infinite expansion. It is a measured-symbolic output of a π-generating procedure, carrying uncertainty and provenance:

$$\pi_\alpha = (v_{\pi,\alpha}, \varepsilon_{\pi,\alpha}, P_\pi).$$

6.2 Constants as Stabilised Generons

A physical or mathematical constant is often written as though it were a fixed point. FSM treats it instead as a stabilised output of a generon. The stability may be metrological, computational, or symbolic.

Definition 6.3 (Measured Constant)**.** *A Measured Constant is a Measured Number whose provenance records a stabilised generative or metrological procedure and whose admissible value remains stable under specified refinement conditions.*

This definition allows the same nominal constant to appear in multiple provenance forms. A computed π_α, a geometrically measured π_α, and an algorithmically generated π_α may be equivalent under a specified collapse relation, but they are not identical before that relation is applied.

6.3 Constants and Attractors

A constant may also be treated as an attractor of repeated symbolic generation. Let $\mathcal{G}_C$ be a constant-generating rule and let m_n be its measured output at stage n. If the trajectory satisfies

$$m_{n+1} \approx_\alpha m_n$$

for sufficiently large n, then the measured constant is the terminal attractor of that generon at resolution α:

$$C_\alpha = \mathrm{Term}_\alpha(\mathcal{G}_C).$$

This provides a bridge between numerical constants and symbolic dynamics.

Chapter 7

Provenance, Equivalence, and Collapse

7.1 Why Value Is Not Enough

Classical equality compares values. FSM compares measured-symbolic objects. Since a measured object carries value, uncertainty, and provenance, equality cannot be reduced to nominal agreement.

Let

$$m_1 = (v_1, \varepsilon_1, P_1), \qquad m_2 = (v_2, \varepsilon_2, P_2).$$

The value components may overlap while the provenance components differ. Conversely, provenance may be equivalent while nominal outputs differ within allowed uncertainty.

7.2 Containment Equivalence

The simplest measured equivalence relation is containment equivalence:

$$m_1 \approx_\delta m_2 \iff |v_1 - v_2| < \varepsilon_1 + \varepsilon_2 + \delta.$$

This tests overlap of measured intervals or containment regions. It does not by itself collapse provenance.

7.3 Provenance Equivalence

Definition 7.1 (Provenance Equivalence). *Let Π be a specified provenance-collapse policy. Two provenance records $P_1, P_2 \in \mathcal{P}$ are provenance-equivalent under Π, written*

$$P_1 \equiv_\Pi P_2,$$

if Π admits them as equivalent generative histories for the purpose at hand.

Definition 7.2 (Full Measured Equivalence). *Two Measured Numbers $m_1 = (v_1, \varepsilon_1, P_1)$ and $m_2 = (v_2, \varepsilon_2, P_2)$ are fully equivalent under (α, Π), written*

$$m_1 \equiv_{\alpha,\Pi} m_2,$$

if

$$m_1 \approx_\alpha m_2$$

and

$$P_1 \equiv_\Pi P_2.$$

7.4 Collapse Relations

Classical mathematics often collapses provenance automatically. FSM makes such collapse explicit.

Definition 7.3 (Value Projection)**.** *The value projection is*

$$\mathrm{Val}(v, \varepsilon, P) = v.$$

It discards uncertainty and provenance.

Definition 7.4 (Classical Collapse)**.** *The classical collapse of a measured-number space is the idealised projection obtained by taking*

$$\varepsilon \to 0,$$

discarding provenance, and treating nominal values as exact.

The classical collapse is useful. It is also lossy. The purpose of FSM is not to forbid collapse but to record what is lost when collapse is performed.

7.5 Provenance Classes

The measured-number classes may be summarised as:

Class	Symbol	Generative provenance
Measured Naturals	$\mathcal{M}_N$	counting / successor generons
Measured Integers	$\mathcal{M}_Z$	signed counting generons
Measured Rationals	$\mathcal{M}_Q$	ratio / division generons
Measured Irrationals	$\mathcal{M}_I$	irrational generons
Measured Transcendentals	$\mathcal{M}_T$	transcendental generons
Measured Constants	$\mathcal{M}_C$	stabilised constant generons

The table is not exhaustive. It is a first taxonomy.

Part III

Measured Sets and Geometric Structures

Chapter 8

Measured Sets and Alphonic Membership

8.1 From Numbers to Sets

If numbers are measured, then sets must also be measured. A classical set is usually treated extensionally: it is determined by its members. FSM cannot begin there, because membership itself is a symbolic distinction requiring finite admissibility.

The prior question is not simply:

$$x \in S?$$

The prior question is:

> By what finite symbolic procedure is x admitted as a member of S, under what uncertainty, and with what provenance?

8.2 Measured Sets

Definition 8.1 (Measured Set)**.** *A Measured Set is a triple*

$$S_M = (E, \varepsilon_S, P_S),$$

where E is a finite collection of admissible measured-symbolic elements, ε_S is the boundary or membership uncertainty, and P_S is the provenance of the set construction.

At fixed Alphon and bounded container, one may define

$$\mathsf{Set}_{\alpha,B}(\mathcal{M}) = \{S_M = (E, \varepsilon_S, P_S) : E \subseteq \mathcal{M}_{\alpha,B},\ |E| < \infty\}.$$

Since $\mathcal{M}_{\alpha,B}$ is finite, each realised measured set is finite.

8.3 Alphonic Membership

Definition 8.2 (Alphonic Membership)**.** *Let $x \in \mathcal{M}$ and let $S_M = (E, \varepsilon_S, P_S)$ be a Measured Set. Alphonic membership is written*

$$x \in_\alpha S_M.$$

It holds when there exists $e \in E$ such that x is containment-equivalent to e at the relevant resolution and the provenance rules of S_M admit the membership relation.

A simple value-based version is

$$x \in_\alpha S_M \iff \exists e \in E \text{ such that } x \approx_\alpha e.$$

A fuller version includes provenance admissibility:

$$x \in_\alpha S_M \iff \exists e \in E : \; x \approx_\alpha e \text{ and } \operatorname{Prov}(x, e, S_M) \text{ is admissible.}$$

8.4 Boundary Membership

Classical set membership is binary. FSM membership may be stable, unstable, or boundary-indeterminate depending on Alphonic resolution.

Definition 8.3 (Boundary Member)**.** *An element x is a boundary member of S_M if small admissible perturbations of x, S_M, or the Alphonic tolerance change the membership outcome.*

Boundary membership is not a defect. It is the ordinary condition of finite symbolic systems near a distinction threshold.

8.5 Measured Union and Intersection

Let

$$S_1 = (E_1, \varepsilon_1, P_1), \qquad S_2 = (E_2, \varepsilon_2, P_2).$$

A provisional measured union may be defined as

$$S_1 \cup_\alpha S_2 = (E_1 \cup E_2, \varepsilon_\cup, P_1 \oplus P_2),$$

where $\varepsilon_\cup$ is determined by the chosen uncertainty-composition rule.

A provisional measured intersection may be defined as

$$S_1 \cap_\alpha S_2 = (E_\cap, \varepsilon_\cap, P_1 \oplus P_2),$$

where

$$E_\cap = \{x \in E_1 : \exists y \in E_2 \text{ with } x \approx_\alpha y\}.$$

These definitions are intentionally provisional. They show the direction: set operations are measured operations with uncertainty propagation and provenance composition.

Chapter 9

Measured Functions, Relations, and Mappings

9.1 Functions as Admissible Transformations

A classical function is often introduced as a relation assigning each input exactly one output. FSM asks a prior question: by what finite procedure is the mapping produced, represented, and admitted?

Definition 9.1 (Measured Function)**.** *A Measured Function from a Measured Set S_M to a Measured Set T_M is a triple*

$$f_M = (R_f, \varepsilon_f, P_f),$$

where $R_f \subseteq S_M \times T_M$ is a finite measured relation, ε_f is the mapping uncertainty, and P_f is the provenance of the mapping rule.

9.2 Approximate Functionality

Exact single-valuedness is a classical idealisation. In finite systems, functionality must be tolerance-bound.

Definition 9.2 (Alphonic Functionality)**.** *A measured relation f_M is Alphoncially functional if whenever*

$$(x, y_1), (x, y_2) \in R_f,$$

then

$$y_1 \approx_\alpha y_2.$$

Thus multiple outputs may be present at the raw symbolic level, but they collapse to a single admissible output under Alphonic equivalence.

9.3 Composition

Let

$$f_M : S_M \to T_M, \qquad g_M : T_M \to U_M.$$

The measured composition is

$$g_M \circ_\alpha f_M = (R_{g\circ f}, \varepsilon_{g\circ f}, P_f \oplus P_g),$$

where

$$R_{g\circ f} = \{(x, z) : \exists y \in T_M \text{ with } (x, y) \in R_f \text{ and } (y, z) \in R_g\}.$$

The uncertainty $\varepsilon_{g\circ f}$ must include propagated uncertainty from both mappings and any additional uncertainty introduced by the intermediate membership test.

9.4 Measured Relations

Measured relations are more primitive than measured functions. A relation may be multi-valued, boundary-sensitive, or provenance-dependent.

This is important for physics, language, and computation. Many real symbolic processes are not functions in the classical sense until a collapse rule is imposed.

Chapter 10

Spherical Uncertainty Distributions as Measured Set Geometry

10.1 The SUD as Geometric Instantiation

The Spherical Uncertainty Distribution (SUD) is the geometric instantiation of a Measured Number. If

$$m = (v, \varepsilon, P),$$

then the uncertainty is not merely an annotation. It is a finite containment region.

A simplified three-dimensional form is

$$\mathrm{SUD}(v, \varepsilon) = \{x \in \mathbb{R}^3 : \|x - v\| \leq \varepsilon\}.$$

This expression should not be read as a return to an ideal continuum. Rather, it is a geometric model of the finite uncertainty region associated with a measured-symbolic object.

10.2 SUD as Measured Set

A SUD can be treated as a Measured Set:

$$S_{\mathrm{SUD}} = (E_{\mathrm{SUD}}, \varepsilon_{\mathrm{SUD}}, P_{\mathrm{SUD}}),$$

where E_{SUD} is the finite collection of distinguishable containment cells or voxels admitted at the specified Alphon.

At fixed α and bounded container B, the SUD is finite:

$$|E_{\mathrm{SUD}}| < \infty.$$

10.3 Packing, Overlap, and Density

When multiple measured symbols occupy a shared symbolic or physical space, their SUDs may overlap, pack, compress, or redistribute. This suggests a finite measured geometry of symbolic interaction.

Let

$$\mathcal{D} = \{\mathrm{SUD}(m_i) : i = 1, \ldots, N\}$$

be a finite SUD family. We may define an overlap graph $\Gamma_\alpha(\mathcal{D})$, where vertices are SUDs and edges indicate Alphonic overlap:

$$\mathrm{SUD}(m_i) \sim_\alpha \mathrm{SUD}(m_j) \iff m_i \approx_\alpha m_j.$$

This graph is a finite measured structure. It may represent numerical ambiguity, symbolic closeness, semantic proximity, measurement conflict, or physical interaction depending on provenance.

10.4 SUD Is Not Automatically a Group

It is tempting to ask whether the SUD is a set, group, field, topology, or measure space. The cautious answer is that the SUD is first a finite uncertainty-containment geometry. Algebraic structures arise only after operations are specified.

The transformations of SUDs may form:

- a monoid, when transformations compose and have identity but not all inverses;
- a group, when transformations are reversible under admissible constraints;
- a groupoid, when transformations are locally reversible or only partially composable;
- a lattice, when containment and refinement dominate;
- a metric space, when distance between containment regions is primary;
- a category, when objects and admissible transformations are foregrounded.

The SUD itself should therefore be introduced as a measured geometric object. Its algebra is induced by the transformations one permits.

Chapter 11

Transformation Structures: Monoids, Groupoids, Groups, and Lattices

11.1 Transformations of Measured Structures

Let S_M be a measured structure. A transformation of S_M is an admissible measured mapping

$$T : S_M \to S_M.$$

The set of such transformations is denoted

$$\mathrm{End}_\alpha(S_M).$$

Composition gives

$$T_2 \circ_\alpha T_1.$$

11.2 Measured Monoids

Definition 11.1 (Measured Monoid)**.** *A Measured Monoid is a measured set M_S equipped with an admissible binary operation $\star_\alpha$ and identity element e_α such that closure, associativity, and identity hold up to Alphonic equivalence.*

The conditions are:

$$a \star_\alpha b \in_\alpha M_S,$$
$$(a \star_\alpha b) \star_\alpha c \approx_\alpha a \star_\alpha (b \star_\alpha c),$$
$$e_\alpha \star_\alpha a \approx_\alpha a \approx_\alpha a \star_\alpha e_\alpha.$$

11.3 Measured Groups

Definition 11.2 (Measured Group)**.** *A Measured Group is a Measured Monoid in which every element has an Alphonic inverse. That is, for every $a \in_\alpha G_M$, there exists $a_\alpha^{-1} \in_\alpha G_M$ such that*

$$a \star_\alpha a_\alpha^{-1} \approx_\alpha e_\alpha$$

and

$$a_\alpha^{-1} \star_\alpha a \approx_\alpha e_\alpha.$$

Measured groups are delicate because reversibility may fail at finite resolution. This is not a weakness. It is the measured counterpart of the Arrow of Finity: finite processes often lose information.

11.4 Measured Groupoids

A groupoid may be more natural than a group for many FSM structures. Local transformations may be reversible within a region while global reversibility fails.

Definition 11.3 (Measured Groupoid)**.** *A Measured Groupoid is a collection of measured objects and admissible partial transformations between them, where each transformation has a local Alphonic inverse on its domain of admissibility.*

This may be especially suitable for SUD transformations, symbolic embeddings, local coordinate changes, and finite measurement frames.

11.5 Measured Lattices

Containment and refinement suggest lattice-like structures. If S_1 and S_2 are measured sets, one can define provisional meet and join operations:

$$S_1 \wedge_\alpha S_2 = S_1 \cap_\alpha S_2,$$

$$S_1 \vee_\alpha S_2 = S_1 \cup_\alpha S_2.$$

Associativity, commutativity, absorption, and distributivity must be checked under Alphonic equivalence rather than assumed exactly.

Part IV

Dynamics, Embedding, and Convergence

Chapter 12

Arithmetic Generons as Finite Symbolic Dynamical Systems

12.1 Operations as Systems

The Finite-Symbol Embedding Theorem applies to finite symbolic dynamical systems. Arithmetic operations can be treated as such systems when unfolded.

Let Ω be an operation such as addition, multiplication, division, exponentiation, root extraction, or constant generation. Let S_Ω be the finite representable state space of the computation implementing Ω. Let

$$T_\Omega : S_\Omega \to S_\Omega$$

be the update rule of the procedure.

Then the operation generates the trajectory

$$\mathcal{T}_\Omega(s_0) = \left(s_0, T_\Omega(s_0), T_\Omega^2(s_0), \ldots, T_\Omega^N(s_0)\right).$$

The compressed notation $\Omega(x)$ is the symbolic projection of this trajectory.

12.2 Root Extraction Example

For the square-root procedure,

$$r_{n+1} = \frac{1}{2}\left(r_n + \frac{x}{r_n}\right),$$

the state s_n may include the current estimate r_n, the input x, the precision parameter, the stopping rule, and the provenance of arithmetic operations used at stage n.

At finite precision, the trajectory terminates when

$$r_{n+1} \approx_\alpha r_n.$$

The output is

$$\sqrt{x}_\alpha = \mathrm{Term}_\alpha(T_{\sqrt{\ }}, r_0).$$

12.3 Decimal Generons

A decimal expansion is also an arithmetic generon. It produces a sequence of finite symbolic states:

$$d_0, d_1, d_2, \ldots, d_N.$$

The classical infinite expansion is the useful-fiction direction. The realised decimal is the terminal finite string admitted by the container.

12.4 Proofs as Symbolic Dynamical Systems

A proof may likewise be treated as a finite symbolic trajectory:

$$\pi = (S_0, S_1, S_2, \ldots, S_N),$$

where each S_i is a symbolic state and each transition is admitted by a rule. The conclusion is not merely a proposition. It is the terminal state of a stability-preserving symbolic path.

This connects arithmetic, logic, and dynamics under the same measured-symbolic framework.

Chapter 13

FSET and the Reconstruction of Computation

13.1 FSET in the Present Volume

FSET replaces classical smoothness, diffeomorphism, and infinite precision with finite representability, bounded observational uncertainty, and geometric stability. This volume extends that view inward: not only external systems such as integer recurrences, cellular automata, and formal language dynamics, but arithmetic operations themselves may be reconstructed as finite symbolic trajectories.

Given a finite observable

$$h : S_\Omega \to \mathbb{R}$$

with observation uncertainty $\varepsilon > 0$, define delay vectors

$$X_k^{(m,\tau)} = \big(\tilde{h}(s_k), \tilde{h}(s_{k-\tau}), \dots, \tilde{h}(s_{k-(m-1)\tau})\big).$$

The reconstructed computation is the point cloud or finite-width trajectory in embedding space generated by these delay vectors.

13.2 The Geometry of a Calculation

A calculation is usually imagined as a sequence of symbols leading to a result. FSET suggests that the calculation has a reconstructible geometry. If the operation is information-rich and non-degenerate, the delay embedding may recover stable structure associated with the computation.

This gives a new interpretation of numerical algorithms:

> An algorithm is not merely a route to a value. It is a finite trajectory through symbolic phase space whose geometry may be reconstructed from observations.

13.3 Computational Attractors

An iterative computation may converge toward an attractor. In classical mathematics this is usually described as convergence to a limit. In FSM, the attractor is the stabilised measured-symbolic output of a finite procedure.

Let A_m be the reconstructed attractor at embedding dimension m. A computation is geometrically stable when there exists m^* and A^* such that

$$d_{\mathrm{H}}(A_m, A^*) \to 0$$

for $m \to m^*$ under the relevant finite reconstruction conditions.

13.4 Open Computational Programme

This suggests several concrete investigations:

1. Reconstruct the phase geometry of square-root algorithms.
2. Compare Newton, Babylonian, bisection, and continued-fraction generons for the same classical target.
3. Examine whether different algorithms produce distinguishable provenance geometries even when their terminal values are Alphoncially equivalent.
4. Analyse decimal expansions of $\sqrt{2}$, π, and e as finite symbolic trajectories.
5. Study proof paths as embedded symbolic trajectories.

Chapter 14

Alphonic Convergence and Dimensional Stabilisation

14.1 The Problem of Dimension

Classical mathematics often treats dimension as a property of an ideal object. FSM treats dimension as the stable measured geometry of a generonic trajectory under finite resolution.

A system may not present its dimensional structure directly. It may have to be unfolded, embedded, measured, and stabilised.

14.2 Dimensional Stabilisation

Definition 14.1 (Dimensional Description)**.** *Let $\mathcal{G}$ be a generon and let α be a measurement resolution. A dimensional description $D_\alpha(\mathcal{G})$ is the inferred finite-dimensional structure associated with the measured trajectory of $\mathcal{G}$ at resolution α.*

Definition 14.2 (Alphonic Dimensional Stabilisation)**.** *A generon $\mathcal{G}$ exhibits Alphonic Dimensional Stabilisation if there exists a stable dimension class D^* such that, under admissible refinement of α,*

$$D_{\alpha_k}(\mathcal{G}) \to D^*$$

within the tolerance of the measurement framework.

This should not be read as convergence to a metaphysical completed dimension. It is convergence to a stable measured-dimensional description.

14.3 Hausdorff Stabilisation

If the system is represented by reconstructed attractors A_{α_k}, one may express stabilisation geometrically:

$$d_{\mathrm{H}}(A_{\alpha_k}, A^*) < \eta$$

for sufficiently refined α_k, where η is an admissible tolerance.

This connects directly with FSET-style attractor convergence. The same structure appears again: finite observations, delay reconstruction, stabilisation under dimension, and terminal geometry.

14.4 The Alphonic Limit as Dimensional Boundary

The Alphonic Limit is not only a numerical resolution floor. It is also a dimensional boundary. Below the Alphonic Limit, further directional distinctions cannot be stabilised. At that boundary, uncertainty becomes isotropic and the SUD becomes the natural containment geometry.

This suggests the following conjecture.

Conjecture 14.1 (Dimensional Convergence Conjecture)**.** *In a finite measured-symbolic system, admissible dimensional descriptions converge toward stable structures at the Alphonic Limit. These structures are represented by finite uncertainty-containment geometries, with the SUD as the primary local form.*

Chapter 15

Physics as Stabilised Measured Structure

15.1 From Symbolic Dynamics to Physics

The bridge into physics begins when exogenous interaction is converted into endogenous symbol through measurement. A physical law is then a stabilised symbolic relation among measured structures.

FSM does not begin with a claim about what reality is beneath measurement. It begins with the claim that formal analysis enters only through finite measured symbols.

15.2 Measured Dimensions in Physics

Physical dimensions may be treated as stable measured structures rather than as primitive background categories. Length, time, mass, charge, and derived dimensions are not merely labels. They are stabilised symbolic outcomes of metrological generons.

A dimensional quantity may be written

$$Q_M = (v, \varepsilon, P, D),$$

where D is the dimensional provenance or dimensional class associated with the measurement.

At sufficient stabilisation, quantities of the same dimension compose. Near boundary regimes, dimensional stability may fail, bifurcate, or require a richer measured structure.

15.3 Convergence to Dimensions at the Alphonic Limit

A possible physical reading of dimensional stabilisation is this:

> The dimensions we measure are the stable attractors of repeated generonic measurement at finite resolution.

If so, then dimensional analysis is not merely a bookkeeping convention. It is a record of stabilised measurement geometry.

This may explain why certain physical relationships become clearer near limiting regimes. As systems are pushed toward the Alphonic boundary of the available measurement apparatus or symbolic framework, hidden compression becomes exposed. The measured trajectory begins to reveal the geometry that the classical expression had concealed.

15.4 Toward Finite Symbolic Physics

The present volume does not attempt to derive a full physics. It prepares the symbolic and mathematical scaffolding needed for such work.

The key future questions are:

1. How are physical dimensions generated and stabilised by measurement procedures?
2. Can SUD packing and overlap model finite uncertainty in physical interaction?
3. Can constants be reinterpreted as measured attractors of metrological generons?
4. Can limiting physical regimes be analysed as failures of classical symbolic compression?
5. Can FSET-style reconstruction reveal hidden structure in physical measurement sequences?

Part V

Working Synthesis

Chapter 16

A Provisional Formal Core

16.1 Core Objects

The following objects provide a provisional formal core for FSM Part II.

Object	Meaning
$\mathcal{M}$	Space of Measured Numbers
$m = (v, \varepsilon, P)$	Measured Number: value, uncertainty, provenance
$\mathcal{M}_{\alpha,B}$	finite measured-number container at Alphon α in bounded container B
$\mathfrak{G}_X$	generon class associated with measured class X
$\mathcal{M}_X$	provenance class of Measured Numbers generated by $\mathfrak{G}_X$
$\mathrm{Term}_\alpha(\mathcal{G})$	Alphonic termination of generon $\mathcal{G}$
$S_M = (E, \varepsilon_S, P_S)$	Measured Set
$x \in_\alpha S_M$	Alphonic membership
$f_M = (R_f, \varepsilon_f, P_f)$	Measured Function or mapping relation
$\mathrm{SUD}(v, \varepsilon)$	Spherical Uncertainty Distribution associated with measured symbol
$D_\alpha(\mathcal{G})$	dimensional description of generon $\mathcal{G}$ at resolution α

16.2 Core Principles

Principle 16.1 (Measured Structure Principle). *Every admissible mathematical structure is represented by finite measured-symbolic objects carrying uncertainty, provenance, and admissibility conditions.*

Principle 16.2 (Compressed Trajectory Principle). *Every mature mathematical notation compresses an unfolded symbolic trajectory.*

Principle 16.3 (Alphonic Termination Principle). *Every realised symbolic trajectory terminates when further unfolding fails to produce a distinguishable state at the relevant Alphonic resolution.*

Principle 16.4 (Provenance Classification Principle). *Measured-number classes are determined by generative provenance, not by nominal value alone.*

Principle 16.5 (Finite Container Principle). *At fixed Alphon and within a bounded representational container, the set of admissible measured-symbolic states is finite.*

Principle 16.6 (Dimensional Stabilisation Principle)**.** *Dimension is the stabilised measured geometry of a symbolic or physical trajectory under finite resolution.*

Chapter 17

Research Directions

17.1 Measured Number Theory

A future measured number theory would investigate the algebra, equivalence, ordering, and convergence of $\mathcal{M}$ and its provenance classes. Key problems include:

1. Define ordering on Measured Numbers with overlapping uncertainty.
2. Develop arithmetic closure rules for $\mathcal{M}_N, \mathcal{M}_Z, \mathcal{M}_Q, \mathcal{M}_I, \mathcal{M}_T$.
3. Classify generons by termination behaviour.
4. Study collisions between provenance classes at finite Alphon.
5. Develop a theory of measured primes and measured divisibility.

17.2 Measured Set Theory

A future measured set theory would investigate membership, union, intersection, complement, functions, equivalence classes, quotient structures, and power-set analogues under finite containment.

Open problems include:

1. Define the measured power set of S_M.
2. Determine when measured union and intersection form a lattice.
3. Develop boundary membership calculus.
4. Study measured cardinality and counting with uncertainty.
5. Connect measured sets to SUD packing geometries.

17.3 Arithmetic Dynamics and FSET

Arithmetic generons may be studied using FSET-style reconstruction.

Possible experiments:

1. Delay-embed the Babylonian square-root algorithm.
2. Compare phase portraits of different square-root algorithms.
3. Delay-embed decimal expansions of $\sqrt{2}$, π, and e.
4. Study proof trajectories as symbolic attractors.
5. Analyse measured irrational generation as convergence to finite terminal attractors.

17.4 Physics and Dimensional Stabilisation

The physics programme asks whether physical dimensions and constants may be understood as stabilised measured structures.

Open problems include:

1. Develop measured dimensional analysis.
2. Model constants as metrological generon attractors.
3. Relate SUD geometry to physical uncertainty and interaction density.
4. Investigate convergence of physical descriptions at the Alphonic Limit.
5. Examine whether classical singularities are failures of symbolic compression.

Chapter 18

Closing Statement

The first volume of Finite Symbolic Mechanics begins from the finite symbol. This second volume begins from the finite trajectory hidden inside the symbol.

The measured number was the first step:

$$m = (v, \varepsilon, P).$$

The present volume extends that step outward. Operations are measured trajectories. Irrationals are Alphonic terminations of irrational generons. Sets are finite containment structures. Membership is admissibility under uncertainty. Functions are measured mappings. The SUD is the local geometry of finite uncertainty. Dimension is a stabilised measured structure. Classical mathematics remains as a powerful collapse, but it is no longer allowed to hide the finite process that made it possible.

The hinge is simple:

> The classical object names the ideal direction. The FSM object is the finite measured-symbolic output of the trajectory.

This is the beginning of measured structures.

Appendix A

Notation Summary

Symbol	Meaning
$\mathcal{M}$	Space of Measured Numbers
$m = (v, \varepsilon, P)$	Measured Number with nominal value v, uncertainty ε, provenance P
ε	strictly positive uncertainty
P	provenance record
$\mathcal{P}$	provenance monoid
α	Alphon or Alphonic resolution context
B	bounded representational container
$\mathcal{M}_{\alpha,B}$	finite set of admissible Measured Numbers in (α, B)
$\mathfrak{G}_N$	counting / successor generon class
$\mathfrak{G}_Z$	signed counting generon class
$\mathfrak{G}_Q$	ratio / division generon class
$\mathfrak{G}_I$	irrational generon class
$\mathfrak{G}_T$	transcendental generon class
$\mathcal{M}_N, \mathcal{M}_Z, \mathcal{M}_Q, \mathcal{M}_I, \mathcal{M}_T$	measured-number provenance classes
$\mathrm{Term}_\alpha(\mathcal{G})$	Alphonic termination of generon $\mathcal{G}$
$\approx_\alpha$	approximate or containment equivalence at Alphonic resolution
$\in_\alpha$	Alphonic membership
$S_M = (E, \varepsilon_S, P_S)$	Measured Set
$f_M = (R_f, \varepsilon_f, P_f)$	Measured Function or measured mapping relation
$\mathrm{SUD}(v, \varepsilon)$	Spherical Uncertainty Distribution
d_H	Hausdorff distance
$D_\alpha(\mathcal{G})$	dimensional description of $\mathcal{G}$ at α

Appendix B

Suggested Insertions into FSM Part I

The following insertion plan is recommended for the existing FSM monograph.

Insertion Point 1: After the Space of Measured Numbers

Insert a chapter titled:

Compressed Operations and Measured Number Classes

This chapter should introduce:

- operations as compressed trajectories;
- Alphonic termination;
- the square-root generon;
- Measured Irrationals;
- finite measured-number containers $\mathcal{M}_{\alpha,B}$;
- provenance classes $\mathcal{M}_N, \mathcal{M}_Z, \mathcal{M}_Q, \mathcal{M}_I, \mathcal{M}_T$.

Insertion Point 2: After the Abacus as Archetype

Insert a subsection titled:

The Abacus and the Termination of Number

This subsection should state that the abacus does not merely represent arithmetic; it performs finite arithmetic. The same principle applies to square-root extraction, decimal expansion, proof generation, and computation.

Insertion Point 3: In the SUD Chapter

Add a note that the SUD is not merely a distribution but the local measured-set geometry of a finite symbol. Avoid prematurely calling the SUD a group. Instead, state that transformations of SUDs may induce monoids, groupoids, groups, lattices, metric spaces, or categories depending on the operation.

Appendix C

Suggested Addition to the FSET Paper

A short section may be added after the definition of finite symbolic dynamical systems or after the definition of trajectory.

Arithmetic Operations as Symbolic Trajectories

Let E be a finite symbolic expression and let Ω be an operation appearing in E, such as multiplication, division, exponentiation, or root extraction. In classical notation, Ω is normally treated as an instantaneous formal operator. In the finite-symbol setting it is instead represented by an update rule

$$T_\Omega : S_\Omega \to S_\Omega,$$

where S_Ω is the finitely representable state space of the computation or symbolic procedure implementing Ω.

The unfolded operation is therefore the trajectory

$$\mathcal{T}_\Omega(s_0) = \left(s_0, T_\Omega(s_0), T_\Omega^2(s_0), \ldots, T_\Omega^N(s_0)\right),$$

where N is finite whenever the procedure terminates at the prescribed finite precision.

For a finite observable $h : S_\Omega \to \mathbb{R}$ with precision $\varepsilon > 0$, this trajectory admits delay vectors and therefore falls directly within the scope of FSET whenever the corresponding non-degeneracy and boundedness conditions are satisfied.

Example: The Square-Root Generon

Consider the square-root procedure for $x = 2$, implemented for example by

$$r_{n+1} = \frac{1}{2}\left(r_n + \frac{2}{r_n}\right).$$

In classical mathematics the sequence is said to converge to the real number $\sqrt{2}$. In the finite-symbol framework, the sequence is a symbolic trajectory generated by an update rule $T_{\sqrt{\ }}$. At finite precision, the realised computation terminates when

$$r_{n+1} \approx_\alpha r_n.$$

The output is not a completed infinite object. It is the terminal finite state of the trajectory:

$$\sqrt{2}_\alpha = \mathrm{Term}_\alpha(T_{\sqrt{\ }}, r_0).$$

Appendix D

Bibliographic Placeholders

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