

**Mathematics as Lenses:
Geofinitism and the Reconstruction
of Discrete Dynamical Structure**

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Abstract. Mathematics is conventionally presented as a single enterprise with agreed foundations, pursuing absolute truths through formal proof. This paper argues that the reality of mathematical practice is more epistemologically complex and more productive: mathematics is best understood as a *collection of lenses* — carefully crafted frameworks, each defined by explicit foundational assumptions, each illuminating certain structural features while necessarily obscuring others.

Under this view, the dominant frameworks of classical analysis and number theory are not wrong, but they are lenses with specific focal lengths. When applied to systems that violate their founding assumptions — particularly discrete, symbolic, and computationally defined processes — they generate not falsity but *illegibility*: structure that exists but cannot be brought into focus.

We argue that robust mathematical understanding of such systems requires three things: the deliberate selection of an appropriate lens; radical clarity about the assumptions that define it; and the construction of consensus through the *convergence of multiple independent lenses* on the same structural features, rather than through the certificate of a single formal proof. *Geofinitism* is presented as precisely such a lens: a measurement-grounded, finite, geometric framework that makes discrete dynamical systems legible in ways that classical approaches cannot. The Collatz integer iteration process serves throughout as an exemplar of a system that is illegible under classical lenses and becomes geometrically transparent under the Geofinite one. This paper is the philosophical foundation for two companion technical works: the Finite-Symbol Embedding Theorem, which provides the formal apparatus, and an empirical reconstruction of the Collatz attractor, which demonstrates its power.

Keywords: philosophy of mathematics, Geofinitism, mathematical lenses, discrete dynamics, Collatz conjecture, finite symbolic systems, measurement-grounded mathematics, consensus epistemology, useful fiction

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1. Introduction: The Problem of Hidden Structure

The map is not the territory, but a good map, used with awareness of its conventions, is enormously more useful than no map at all. The danger lies in forgetting the conventions.

— after Korzybski

Consider the following sequence, generated from the starting value 27:

27, 82, 41, 124, 62, 31, 94, 47, 142, 71, 214, 107, 322, 161, ...

The generating rule is elementary: halve even numbers, triple odd numbers and add one. The sequence climbs to 9232, crashes, climbs again, and after 111 steps reaches 1, entering the terminal cycle (1, 4, 2). Starting from 27, this is the Collatz process. Despite extraordinary simplicity of definition and more than eighty years of effort, no one has proved that every positive integer eventually reaches 1.

The Collatz conjecture is not merely an open problem. It is, we argue, a *diagnostic* — a system that exposes a systematic mismatch between the mathematical lenses that have been brought to bear on it and the lens that its structure actually requires. Viewed as a one-dimensional integer sequence, the Collatz process is erratic, with apparent irregularity at every scale. Viewed as a trajectory in a reconstructed phase space, all 999 starting values between 2 and 1000 resolve into a single coherent comma-shaped manifold converging toward a unique geometric attractor. The structure was always there; it required a different lens to become visible.

This is not an isolated phenomenon. Complex numbers were once called “fictitious” precisely because the real-number lens could not accommodate their geometry. Non-Euclidean geometry was “impossible” until Gauss, Lobachevsky, and Bolyai recognised that the parallel postulate was an assumption, not a truth, and that removing it opened rather than closed a legitimate mathematical world. Quaternions, transfinite cardinals, category theory — each represented a deliberate lens-shift that rendered previously illegible structure legible.

The thesis of this paper is that discrete, symbolic, and computationally defined dynamical systems are in a structurally analogous position today. The dominant mathematical lenses — classical analysis, number theory, formal logic — are not wrong. They are lenses tuned for other focal lengths. Making discrete systems fully legible requires a new lens with explicitly declared foundational assumptions.

Geofinitism is that lens.

This paper proceeds as follows. Section 2 develops the lens metaphor philosophically, with historical grounding. Section 3 argues that the power of a lens depends entirely

on the clarity and honesty of its defining assumptions. Section 4 develops a philosophy of mathematical consensus grounded in multi-lens convergence rather than single-lens proof. Section 5 defines *Geofinitism* as a deliberate philosophical stance and contrasts it with Platonism and classical formalism. Section 6 shows how this lens makes discrete dynamical systems like Collatz geometrically legible. Section 7 provides a roadmap to the companion technical papers. Section 8 concludes with a call for deliberate lens choice in mathematical practice.

2. Mathematics as a Collection of Lenses

2.1. The Plurality of Mathematical Frameworks

What is a mathematical framework? At minimum, it is a combination of: a *state space* (what objects exist and how they are represented), a collection of *permitted operations* (what can be done to or with those objects), a set of *foundational assumptions* (what is taken as given without proof), and a *notion of validity* (what counts as a correct result).

Different frameworks make different choices at each level, and these choices are not merely technical. They determine which questions can be asked, which structures are visible, and which are invisible by construction.

Real analysis assumes that quantities vary continuously and that limits exist. Within this framework, differentiability is a natural and central concept. Number theory assumes that integers are primary objects and studies their combinatorial and divisibility structure. Within this framework, modular arithmetic and prime factorisation are natural tools. Formal logic assumes that propositions have definite truth values and that proofs are finite deductive sequences. Within this framework, provability and decidability are the central concerns.

Each of these frameworks is a lens. Each has been extraordinarily productive within its domain of applicability. And each, when applied beyond that domain, produces not clear answers but *unresolvable difficulty* — not because the problems are intrinsically hard, but because the lens is misaligned with the structure being examined.

2.2. Historical Lens-Shifts

The history of mathematics can be read, in significant part, as a history of deliberate lens-shifts — moments when a community recognised that a previously intractable structure required a new foundational framework.

Complex numbers. The quadratic formula applied to $x^2 + 1 = 0$ produced $\sqrt{-1}$, which the real-number lens classified as “impossible.” Cardano [Cardano, 1545] called such quantities *fictis* (fictitious). The fiction became indispensable when it

was recognised that the complex plane was the *natural habitat* of polynomial roots: the Fundamental Theorem of Algebra holds over \mathbb{C} but not over \mathbb{R} . The structure was always present; the lens had to change for it to become visible.

Non-Euclidean geometry. The parallel postulate was treated for two millennia as either obviously true or provable from the other axioms. Gauss recognised as early as 1816 that it was neither: it was a lens choice [Gray, 2007]. Removing it did not produce nonsense; it produced hyperbolic and elliptic geometries, which turned out to be physically instantiated in the curvature of spacetime.

Infinitesimals. Newton and Leibniz used infinitely small quantities to build calculus. The classical analysis lens, as crystallised by Cauchy and Weierstrass, eliminated infinitesimals in favour of the ϵ - δ formalism — a genuine change of lens, not a refinement of the same one. Abraham Robinson’s non-standard analysis [Robinson, 1966] later showed that infinitesimals were not incoherent; they required a different foundational framework (hyperreal numbers) in which they were legitimate, finite objects.

Category theory. When Eilenberg and Mac Lane introduced categories [Eilenberg & Mac Lane, 1945], many mathematicians regarded them as abstraction for its own sake. In time, it became clear that category theory was a lens that made previously hidden structural correspondences — between algebra, topology, and logic — simultaneously visible. The Curry-Howard correspondence, the Langlands programme, and homotopy type theory are among the structures revealed by this lens.

In each case, the same pattern recurs: a lens is operative within its domain; a class of problems proves intractable; a deliberate foundational shift opens the structure to analysis. The new lens does not invalidate the old one — real analysis remains correct within its domain — but it makes visible what the old lens could not see.

2.3. What a Lens Conceals

An underappreciated corollary of the lens metaphor is that every lens *necessarily conceals* as well as reveals. The assumptions that give a lens its clarity also determine its blind spots.

Classical analysis conceals the structure of discrete systems by imposing a topology in which integers have no interior — every open set containing an integer contains uncountably many non-integers, so integer-specific structure cannot be isolated. Number theory conceals the geometric and dynamical structure of integer sequences by representing them as combinatorial objects rather than trajectories in phase space. Formal proof theory conceals the *computational* structure of mathematical processes by abstracting away the resource cost of derivations.

These are not defects; they are the price of the lens’s power in its intended domain. The defect arises only when a lens is applied in a domain where its concealment is consequential, and this fact is not acknowledged.

3. The Primacy of Clear Assumptions

3.1. Hidden Assumptions as Epistemological Liabilities

A lens is only as powerful as the clarity and honesty of its defining assumptions. When assumptions are hidden — either because they are regarded as “obvious,” or because the framework predates the tools to state them precisely — the lens becomes an epistemological liability. It applies where it should not, generates intractable problems that dissolve when the assumption is made explicit and dropped, and creates the illusion that difficulty is intrinsic to the problem rather than an artefact of the framework.

A lens is only as powerful as the clarity and honesty of its defining assumptions. Hidden assumptions are not foundations: they are unmarked boundaries that one cannot reason across.

Consider Takens’ delay embedding theorem [Takens, 1981], which is central to the technical companion works. The theorem guarantees that the attractor of a dynamical system can be reconstructed from scalar observations via delay coordinates. Its proof relies on three conditions: smoothness of the evolution rule, existence of an underlying differentiable manifold, and the availability of observations at infinite precision. None of these conditions is stated prominently in typical applications of the theorem. They are assumed as background. When the theorem is applied to integer sequences — as researchers in computational number theory and discrete dynamical systems might naturally wish to do — the result is at best informal and at worst invalid. The theorem’s domain of applicability is not coextensive with the domain of interest.

This is not a criticism of Takens; it is an observation about how foundational assumptions propagate through a field without examination. The *Geofinitism* programme addresses this by making every assumption explicit and replacing each one with a finite-precision, measurement-grounded analogue where necessary.

3.2. The Spectrum from Productive Idealisation to Category Error

Not all hidden assumptions are equally problematic. We propose a distinction between two modes:

Productive idealisation. An assumption that is technically violated by the system

under study but which introduces only quantitatively bounded error. Treating a very long polymer as a continuum is a productive idealisation: the discrete atomic structure introduces corrections of order $1/N$ where N is the chain length, and for large N these are negligible. The idealisation does not change the qualitative structure of the results.

Category error. An assumption that, when violated, changes the *qualitative* structure of what can be said. Applying smooth manifold theory to an integer sequence is not a productive idealisation: integers have no tangent bundle, no notion of derivative, no smooth structure at any scale. The classical Takens theorem applied to a Collatz sequence is not “approximately valid”; it is categorically inapplicable. Results obtained under it are not slightly wrong; they have no definable error bound. The distinction matters because the appropriate response is different in each case. Productive idealisations can be refined into more accurate approximations within the same framework. Category errors require a lens-shift.

3.3. Explicit Assumptions as a Scientific Value

We argue that the practice of making foundational assumptions maximally explicit should be treated as a first-order scientific value, not merely a technical nicety. Explicit assumptions enable:

1. *Domain awareness*: knowing where the lens applies and where it does not.
2. *Deliberate extension*: recognising which assumption must be relaxed to extend the framework to a new class of systems.
3. *Lens comparison*: understanding precisely what is different between two frameworks, and what each gains and loses relative to the other.
4. *Error localisation*: when results are inconsistent, identifying which assumption is the source rather than concluding the system is pathological.

Geofinitism is built on this value. Its Five Pillars — Geometric Container Space, Approximations and Measurements, Dynamic Flow, Useful Fiction, and Finite Reality — are not merely content claims. They are explicit statements of the foundational assumptions under which Geofinite results are obtained. A reader who disagrees with Pillar 5 (Finite Reality) knows exactly where and why they diverge.

4. The Philosophy of Mathematical Consensus

4.1. Proof as a Single-Lens Certificate

In classical mathematical practice, *proof* occupies a privileged epistemological position. A formal proof within an axiomatic system is the accepted standard for

mathematical knowledge. This is not merely a convention; it has deep advantages: proofs are checkable, composable, transferable across contexts, and in principle complete within a given framework.

However, proof is a *single-lens certificate*. It certifies that a statement follows, within a specific axiomatic framework, from specific axioms. It says nothing about whether the framework is the right one for the problem at hand, whether the result would persist under a lens-shift, or whether the structure being proved about corresponds to the structure of the system one actually cares about.

This limitation is not a defect of proof as a tool; it is a natural consequence of proof's power. The specificity that makes a proof rigorous also makes it contingent on its framework.

4.2. Gödel's Shadow: The Formal Limit of Single Lenses

Gödel's incompleteness theorems [Gödel, 1931] establish that any sufficiently powerful consistent formal system contains true statements that cannot be proved within that system. This is not merely a technical result; it is a fundamental epistemological constraint on single-lens mathematics. No formal system, however powerful, is a complete mirror of mathematical truth.

This has a direct implication for multi-lens epistemology: if no single formal framework can prove all truths within its own domain, the case for consulting multiple independent frameworks — for treating consensus across lenses as a form of knowledge in its own right — is not merely pragmatic but formally motivated. When a statement that is unprovable in framework F_1 can be *witnessed geometrically* in framework F_2 and *verified computationally* in framework F_3 , the convergence of F_2 and F_3 is providing evidence that F_1 's proof-gap reflects a limitation of F_1 rather than indeterminacy of the statement itself.

4.3. Convergence of Lenses as Robust Understanding

We propose a positive epistemology to complement the negative Gödelian one. *Robust mathematical understanding* of a system or conjecture is achieved not by a single proof but by the *convergence of multiple independently defined lenses* on the same structural features.

Consensus in mathematics is not monolithic agreement on axioms. It is the convergence of independent evidence: when several carefully defined lenses, each with honest assumptions, each examined in its own terms, point to the same geometric or structural feature, that convergence constitutes knowledge of a kind that a single proof within a single framework cannot provide.

This epistemology has historical precedent. The rotation of Mercury’s perihelion was anomalous within Newtonian mechanics for fifty years before general relativity explained it. During those fifty years, the anomaly was not merely a computational curiosity; it was a convergent signal from multiple observational lenses (optical, spectroscopic, ephemeris) that something was structurally inadequate in the Newtonian framework. The convergence of these independent observations constituted genuine knowledge — not proof that Newton was wrong, but robust evidence that a different lens was needed.

In mathematics, the analogue is this: when an analytic lens, a probabilistic lens, a geometric lens, and a computational lens all point to the same attractor structure for the Collatz process, that convergence is not merely suggestive. It is a form of mathematical understanding that neither the lack of a classical proof, nor the limitation of any individual lens, can diminish.

4.4. Contrast with “Proof or Nothing”

The dominant culture of pure mathematics, particularly in analytic number theory, tends toward a “proof or nothing” epistemology: results that have not been formally proved are not knowledge, and empirical or computational evidence, however extensive, does not raise the epistemic status of a conjecture.

We do not dismiss this standard; for questions of absolute logical necessity, it is correct. But we observe that it creates a systematic bias: it directs attention and resources toward problems that are soluble within the existing framework, and away from problems — like Collatz — that may require a lens-shift before a proof becomes possible. The “proof or nothing” standard, applied too strictly, may actually impede the search for proofs by discouraging the foundational work that makes new classes of problems tractable.

This parallels Lakatos’s observation [Lakatos, 1976] that mathematical research programmes have both a “hard core” of protected assumptions and a “protective belt” of auxiliary hypotheses. When the hard core is never examined, the protective belt accumulates ad hoc modifications, and the research programme progressively loses explanatory power. Lens-shifting is, in Lakatosian terms, the replacement of a degenerating hard core with a progressive one.

5. Geofinitism as a Deliberate Lens

5.1. Definition

Geofinitism is a mathematical philosophy and an associated programme of technical development. It is grounded in five foundational claims, stated here as the explicit

assumptions of the lens:

- P1. Geometric Container Space.** Mathematical structure exists as geometric configuration in a space with definite dimensionality. Abstraction is always abstraction *from* geometry, not the other direction.
- P2. Approximations and Measurements.** All mathematical quantities are inherently approximations, carrying bounded uncertainty. Exact values are useful fictions — productive idealisations for cases where error bounds are negligible — not ontological primitives.
- P3. Dynamic Flow.** Mathematical systems are dynamical: they evolve, and their structure is revealed by the geometry of their evolution. Static objects are snapshots of dynamical processes.
- P4. Useful Fiction.** Classical infinite-object mathematics (\mathbb{R} , Cantorian set theory, smooth manifolds) is neither wrong nor primary. It is *useful fiction*: a productive idealisation that is meaningful and applicable where its foundational assumptions are approximately satisfied, and which must be recognised as fictional — rather than substituted as literal — when applied to inherently finite systems.
- P5. Finite Reality.** The natural domain of mathematics is finite: finite symbolic representations, finite computation, finite measurement. Infinity is a limit process, not a state. Mathematical statements about infinite objects are statements about the behaviour of finite processes as their parameters grow, not statements about literally infinite entities.

5.2. Measurement as Ontology

A crucial distinction separates *Geofinitism* from mere operationalism or instrumentalism. Operationalism holds that mathematical quantities are *defined by* measurement procedures; *Geofinitism* makes a stronger claim.

In the Geofinite framework, measurement is not an epistemological tool for accessing pre-existing mathematical objects. It is the *ontological ground* of mathematical structure: the geometric configurations that measurement reveals are not approximations to ideal Platonic objects; they are the objects. The “true” attractor of the Collatz system is not a perfect set-theoretic object that our finite measurements approximate more or less well. It is the finite-precision geometric structure that converges as measurement precision is refined. Refining the precision does not approach a limit; it *is* the mathematical content.

This distinguishes *Geofinitism* sharply from:

Platonism. Mathematical objects exist independently of minds and measurement in an abstract realm. *Geofinitism* rejects the independent existence of infinite Platonic

objects while retaining the objectivity of geometric structure: structure is objective because it is stable across independent measurements, not because it exists in a non-physical realm.

Formalism. Mathematics is a formal game played with symbols according to rules, with no intrinsic meaning. *Geofinitism* rejects this: the symbols encode geometric relationships, and those relationships have definite meaning grounded in measurement.

Intuitionism. Mathematical objects exist only insofar as they can be mentally constructed. *Geofinitism* shares the finite-construction emphasis but grounds it in physical measurement rather than mental construction, making it intersubjectively accessible.

5.3. The Role of Useful Fiction

Pillar 4 — Useful Fiction — deserves special attention because it is the most likely source of misreading. *Geofinitism* does not propose that classical mathematics should be abandoned. Newton’s calculus, Riemann integration, Lebesgue measure, and the classical theory of smooth dynamical systems are among the most powerful intellectual tools humans have constructed. They remain so.

The claim is more precise: they are useful *fictions* in the specific sense that their foundational assumptions (continuity, infinite divisibility, exact real-valuedness) are approximately satisfied in the physical world up to scales far below human observation, and exactly satisfied in a mathematical idealisation. When both conditions hold, using them as literal descriptions is a productive shorthand.

When a system is *explicitly symbolic and discrete* — as integer sequences are — the useful fiction offers no purchase. There is no continuity to exploit, no manifold to embed in, no smooth structure to differentiate. Using the smooth framework is not a productive idealisation; it is a category error. In this regime, the Geofinite lens is not an alternative to the smooth one; it is the only correctly aligned option.

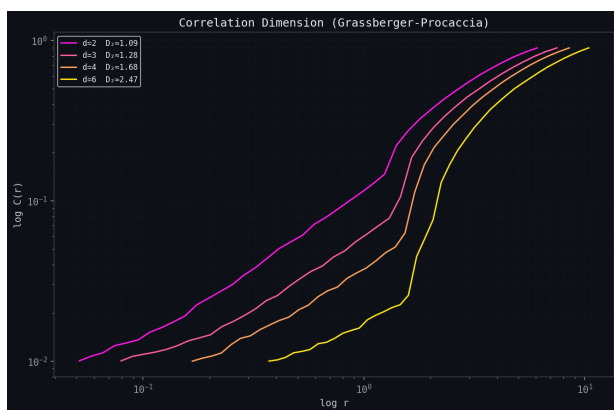
6. Making Discrete Systems Legible

6.1. The Illegibility Problem

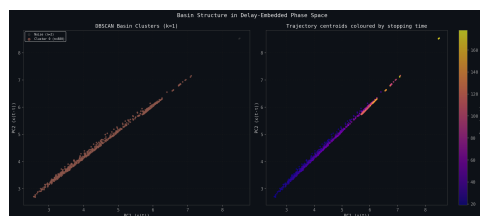
A discrete dynamical system generates a scalar sequence. Examined in one dimension, that sequence may appear erratic, sensitive, and structureless. This appearance is not a property of the system; it is a property of the *projection*. When the state of a higher-dimensional process is observed through a single scalar, the resulting sequence discards dimensional information. The apparent irregularity is the shadow of a higher-dimensional geometric structure projected onto a line.

This point can be made precise. Takens’ delay embedding theorem [Takens, 1981] shows that for smooth systems, the projection loss can be reversed: by constructing delay vectors from the scalar sequence, the original geometric structure can be reconstructed. The Finite-Symbol Embedding Theorem [Haylett, 2026b] — the first companion paper to this work — extends this principle to symbolic systems, showing that the reconstruction is valid under finite-precision, measurement-grounded conditions.

6.2. The Comma Manifold: Collatz Made Visible



(a) Phase portrait of 300 Collatz trajectories in delay-embedded space ($\tau = 1$, $d = 3$, log-normalised), coloured by stopping time. The coherent comma-shaped manifold is invisible in the one-dimensional integer sequence.



(b) DBSCAN clustering of trajectory centroids. A single connected basin contains 99.8% of all trajectories, directly corroborating the single-attractor hypothesis.

Figure 1: The Collatz attractor made visible under the Geofinite lens. Both panels are derived from empirical analysis of 999 trajectories [Haylett, 2026c].

Figure 1 shows what happens when the Geofinite lens is brought to bear on the Collatz process. The left panel displays 300 trajectories as paths in a three-dimensional delay-embedded space, projected onto two coordinates. The right panel shows the result of density-based clustering applied to the trajectory centroids in this space.

Two things are immediately apparent. First, what was a disordered integer sequence becomes, in the reconstructed phase space, a structured geometric object: a comma-shaped manifold through which all trajectories pass, regardless of starting value. Trajectories with long stopping times (warm colours) trace the extended arm of the comma; trajectories with short stopping times (cool colours) enter near the base and converge immediately toward the $(1, 4, 2)$ attractor neighbourhood.

Second, the clustering analysis finds a single connected basin. Of 882 embedded trajectories, 880 — 99.8% — belong to one cluster. The two outliers are numerical boundary effects. This is not a weak or approximate result. It is a direct geometric

statement: in the reconstructed phase space, all Collatz trajectories inhabit a single attractor basin.

Neither of these facts is visible in the one-dimensional integer sequence. They are a property of the system’s geometric structure in the higher-dimensional space. The Geofinite lens, by replacing smoothness assumptions with finite-symbolic ones, makes this structure accessible.

6.3. What the Lens Change Achieves

The transition from classical number-theoretic analysis to Geofinite delay embedding achieves four things simultaneously:

Illegibility to legibility. The erratic one-dimensional sequence becomes a coherent geometric object. The system’s structure is revealed rather than concealed.

Category alignment. The analytical tools (phase-space reconstruction, Lyapunov exponents, recurrence analysis, correlation dimension) are drawn from nonlinear dynamics and are appropriate to the system’s actual structure. Applying them here is not analogical; it is, by virtue of the FSET, formally licensed.

Conjecture reframing. The Collatz conjecture, in classical form, asks: “Does every positive integer eventually reach 1?” Under the Geofinite lens, this becomes: “Does the reconstructed attractor have a single connected basin?” The second question is geometrically tractable in a way the first is not. It admits empirical investigation, admits formal treatment via the FSET, and connects to a body of theory about attractor basins in nonlinear dynamical systems.

Convergent multi-lens evidence. The geometric evidence is not a replacement for classical proof; it is an independent lens converging on the same structure that the classical approach aims to prove. Lyapunov analysis (positive but bounded exponents), recurrence quantification (high determinism > 0.93), correlation dimension (sub-ambient, consistent with a low-dimensional attractor), and clustering (single basin) are four independent analyses, each from a distinct analytical tradition, all pointing to the same conclusion. This is the multi-lens consensus described in Section 4 in action.

7. Roadmap to the Companion Works

This paper is the first of three works that together constitute a complete treatment: philosophical foundation, formal apparatus, and empirical demonstration.

Paper II: The Finite-Symbol Embedding Theorem

Haylett [2026b] develops the formal apparatus that turns the philosophical argument of Section 6 into a rigorous mathematical result. The central theorem — the Finite-Symbol Embedding Theorem (FSET) — proves that any finite symbolic dynamical system satisfying an appropriate non-degeneracy condition admits a delay embedding whose reconstructed geometric structure converges, in the Hausdorff metric on compact sets, to a stable limit attractor as the embedding dimension is increased.

The proof proceeds in three stages. A *Trajectory Separation Lemma* establishes the FSET’s analogue of Takens’ genericity condition: distinct orbits become geometrically distinguishable in sufficiently high embedding dimension, with the required dimension bounded above by the orbit’s first separation time under the observable. A *Geometric Stability Proposition* establishes that the reconstructed attractor is Cauchy in the Hausdorff metric over compact sets (applying the Blaschke selection theorem). The *Main Theorem* assembles these into four claims: injectivity up to measurement precision, geometric stability, attractor convergence, and single-basin uniqueness.

Four corollaries establish: that classical Takens is recovered as the precision parameter $\epsilon \rightarrow 0$ (situating Takens as a special case of the FSET, not vice versa); noise robustness; the finite-precision bound on attractor resolution; and uniqueness of the reconstructed basin for systems with a unique symbolic attractor.

The Collatz system is verified to satisfy all FSET conditions explicitly (Table 1 in Paper II), converting the Collatz conjecture into a precisely stated geometric claim: the Collatz attractor is the unique limit of the FSET embedding, and the conjecture is equivalent to the claim that this attractor has a single connected basin.

Paper III: Empirical Reconstruction of the Collatz Attractor

Haylett [2026c] implements the FSET framework computationally, applying it to 999 Collatz trajectories ($n_0 \in [2, 1000]$). The implementation comprises five modules (sequence generation, delay embedding, nonlinear analysis, visualisation, experiment runner) and follows a systematic protocol covering: average mutual information for delay selection; false nearest neighbours for dimension selection; phase-space reconstruction and phase portraits; Lyapunov exponent estimation via the Rosenstein method; Recurrence Quantification Analysis; Grassberger-Procaccia correlation dimension estimation; and DBSCAN basin clustering.

The results are consistent across all analyses and directly instantiate the theoretical predictions of the FSET:

- Phase portraits reveal the coherent comma-shaped manifold of Figure 1a.
- Lyapunov exponents $\lambda_1 \approx 0.04$ – 0.06 confirm bounded chaos.

- RQA determinism > 0.93 confirms structured, quasi-periodic phase-space visits.
- Correlation dimension $D_2 < d$ at all tested dimensions confirms a low-dimensional invariant set.
- DBSCAN finds a single connected basin containing 99.8% of trajectories.

Taken together, Papers II and III form a self-contained technical programme: the FSET provides the formal licence for the empirical methods, and the empirical results demonstrate the FSET's predictive and explanatory power on a non-trivial, well-known open problem.

The Trilogy as a Research Stance

The three papers together embody a research philosophy: that mathematical progress on intractable problems benefits from explicit foundational reflection (Paper I), a properly grounded theoretical extension (Paper II), and empirical demonstration that the extension is productive rather than merely abstractly possible (Paper III). None of these is sufficient alone. The philosophy without the theorem is ungrounded; the theorem without the empirical application is untested; the empirical results without the theoretical foundation are informal. Together, they constitute a case that the Geofinite lens is not merely philosophically defensible but mathematically productive.

8. Conclusions: A Call for Deliberate Lens Choice

We return to the opening diagnostic. The Collatz conjecture has resisted proof for eighty years. We have argued that this resistance is not evidence that the problem is intrinsically beyond reach, but that the dominant analytical frameworks — the lenses that have been brought to bear on it — are misaligned with its structure. When the correct lens is applied, the structure becomes visible, the appropriate tools become applicable, and the conjecture transforms from an impenetrable number-theoretic puzzle into a tractable geometric question.

The broader claim is that this situation recurs throughout mathematics. There is a class of systems — inherently discrete, symbolic, computational — for which classical smooth and analytic frameworks are not the right lenses. The resistance these systems have shown to classical methods reflects not their intrinsic difficulty but the category mismatch between system and framework. *Geofinitism* is a proposal for the appropriate framework: finite, measurement-grounded, geometric, explicitly bounded.

We close with three calls.

For deliberate lens choice. Before applying analytical machinery to a new class

of problems, ask: are the foundational assumptions of this framework satisfied by this system? Is this a productive idealisation or a category error? If the latter, what lens is required?

For explicit assumptions. State foundational assumptions as the first act of any research programme. Not as disclaimers, but as the load-bearing elements of the framework. Assumptions that cannot be stated clearly are assumptions that cannot be examined, extended, or replaced when necessary.

For multi-lens consensus. Resist the tendency to regard a result as merely preliminary or informal in the absence of a classical proof. When multiple independent lenses, each carefully defined, converge on the same geometric feature, that convergence is a form of mathematical knowledge. Not equivalent to proof within a single formal system, but not subordinate to it either. The two forms of knowledge are complementary, and a research programme that cultivates both is more robust than one that demands only the second.

The structure of mathematics, like the structure of the systems it studies, is not fully visible from any single vantage point. The aim is not to find the one true lens, but to build a collection of well-crafted, honestly specified lenses, and to know when to use which one.

Simul Pariter — Together, Equally.

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