The Attralucian Essays: Exploring the Finite

First Edition

 Typeset in LaTeX

Cite as: Complex Analysis as Takens Embedding:
A Dynamical Systems Foundation for Analytic Functions,
Kevin R. Haylett, February 2026

finitemechanics.com/essays/essay-geometric-numbers-2.pdf

The Attralucian Essays

Complex Analysis as Takens Embedding: A Dynamical Systems Foundation for Analytic Functions

Kevin R. Haylett

Abstract

We present a novel interpretation of classical complex analysis through the lens of dynamical systems theory. By recognizing the Hilbert transform as a specific instance of Takens' time-delay embedding, we demonstrate that analytic functions arise naturally as those dynamical systems for which the embedding is information-preserving. This perspective demystifies the concept of "imaginary" numbers, revealing them as coordinates in an optimal delay coordinate space. We establish formal connections between the Cauchy-Riemann equations, the Bedrosian theorem, and the topological properties of delay embeddings, and show how fundamental results of complex analysis—including the Cauchy integral formula and the Riemann mapping theorem—can be reinterpreted as statements about dynamical systems and their phase space reconstructions.

Introduction

The theory of analytic functions of a complex variable stands as one of the most elegant and powerful edifices in mathematics. Yet for generations of students, the introduction of the "imaginary" unit $i = \sqrt{-1}$ retains an air of mystery. Why should adjoining a formal square root of -1 to the real numbers yield such profound results? Why do analytic functions possess their remarkable properties—conformal mapping, Cauchy's integral theorem, the identity theorem—that have no real-variable analogue?

Concurrently, the theory of dynamical systems has developed powerful tools for reconstructing phase space dynamics from scalar time series. Takens' embedding theorem [1] provides conditions under which delay-coordinate maps preserve the topological structure of the underlying attractor. This paper proposes that these two seemingly disparate frameworks are intimately connected: **complex analysis is precisely the study of dynamical systems under a specific, optimal delay embedding**.

We develop this connection systematically, showing that:

1. The Hilbert transform, which generates the imaginary part of an analytic signal, corresponds to a particular choice of time delay in Takens' construction.

2. The Cauchy-Riemann equations are equivalent to the condition that this delay embedding yields a conformal immersion of the dynamics.

3. The "imaginary" unit i is best understood as the operator that advances a signal by a quarter-period phase shift—the optimal delay for preserving oscillatory information.

4. Fundamental theorems of complex analysis find natural interpretations as statements about dynamical systems and their reconstructions.

Preliminaries

Complex Analysis Fundamentals

We begin by recalling the standard definition of analyticity.

[Analytic Function] A function $f : \mathbb{C} \to \mathbb{C}$ written as $f(z) = u(x, y) + iv(x, y)$ is *analytic* (or holomorphic) on an open set $U \subset \mathbb{C}$ if it is complex-differentiable at every point of U. Equivalently, u and v satisfy the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{1}$$

and have continuous first partial derivatives.

The analytic signal of a real-valued function $x(t)$ is constructed via the Hilbert transform:

[Hilbert Transform] For a function $x(t) \in L^2(\mathbb{R})$, the Hilbert transform $\mathcal{H}[x](t)$ is defined by the principal value integral:

$$\mathcal{H}[x](t) = \frac{1}{\pi}\text{P.V.} \int_{-\infty}^{\infty} \frac{x(\tau)}{t - \tau} d\tau \tag{2}$$

[Analytic Signal] The analytic signal associated with $x(t)$ is:

$$z(t) = x(t) + i\mathcal{H}[x](t) \tag{3}$$

Takens' Embedding Theorem

We now recall the fundamental result from dynamical systems theory that motivates our investigation.

[Takens, 1981] Let M be a compact manifold of dimension m. For a smooth diffeomorphism $\phi : M \to M$ and a smooth observation function $h : M \to \mathbb{R}$, the map $\Phi : M \to \mathbb{R}^{2m+1}$ defined by:

$$\Phi(x) = (h(x), h(\phi(x)), h(\phi^2(x)), \ldots, h(\phi^{2m}(x))) \quad (4)$$

is generically an embedding (a diffeomorphism onto its image).

For continuous-time systems with flow ϕ_t, the discrete sampling becomes:

$$\Phi(x) = (h(x), h(\phi_\tau(x)), h(\phi_{2\tau}(x)), \ldots, h(\phi_{2m\tau}(x))) \quad (5)$$

where $\tau > 0$ is a suitable time delay.

The Hilbert Transform as an Optimal Delay Embedding

Our central observation is that the Hilbert transform can be viewed as a particular infinite-dimensional delay embedding with a specific weighting function.

[Delay Embedding Operator] For a function $x : \mathbb{R} \to$

$\mathbb{R}$ and a delay $\tau > 0$, define the 2-dimensional delay embedding:

$$\Phi_\tau[x](t) = (x(t), x(t-\tau)) \in \mathbb{R}^2 \tag{6}$$

[Hilbert Embedding] Define the Hilbert embedding operator:

$$\Phi_{\mathcal{H}}[x](t) = (x(t), \mathcal{H}[x](t)) \in \mathbb{R}^2 \tag{7}$$

[Hilbert Transform as Integral of Delays] The Hilbert transform admits the representation:

$$\mathcal{H}[x](t) = \frac{1}{\pi}\int_0^\infty \frac{x(t-\tau) - x(t+\tau)}{\tau} d\tau \tag{8}$$

Starting from (2), we split the integral:

$$\begin{aligned}\mathcal{H}[x](t) &= \frac{1}{\pi}\lim_{\varepsilon\to 0}\left(\int_{-\infty}^{t-\varepsilon} \frac{x(\tau)}{t-\tau} d\tau + \int_{t+\varepsilon}^{\infty} \frac{x(\tau)}{t-\tau} d\tau\right) \\ &= \frac{1}{\pi}\int_0^\infty \frac{x(t-\tau) - x(t+\tau)}{\tau} d\tau\end{aligned}$$

after the substitution $\tau = t - u$ in the first integral and $\tau = u - t$ in the second.

This representation reveals that $\mathcal{H}[x](t)$ is a weighted average of **all** delay coordinates $x(t-\tau)$ and advanced coordinates $x(t+\tau)$, with weight $1/\tau$. Unlike Takens' discrete embedding which samples at specific multiples of τ, the Hilbert transform uses a continuum of delays with a harmonic weighting.

Analyticity as an Embedding Condition

We now establish the central result: the Cauchy-Riemann equations are precisely the condition that the Hilbert embedding preserves local geometric structure.

[Analyticity and Conformal Embedding] Let $x(t)$ be a real-analytic function and define its Hilbert embedding $\Phi_{\mathcal{H}}[x](t) = (u(t), v(t))$ where $u(t) = x(t)$ and $v(t) = \mathcal{H}[x](t)$. Then the curve traced by $\Phi_{\mathcal{H}}[x]$ in $\mathbb{R}^2$ is the image of an analytic function $f(t) = u(t) + iv(t)$ if and only if the following condition holds:

$$\frac{du}{dt} = \frac{dv}{d\hat{t}} \quad \text{and} \quad \frac{du}{d\hat{t}} = -\frac{dv}{dt} \tag{9}$$

where $\hat{t}$ denotes the coordinate along the normal direction to the curve.

Consider the complex function $f(t) = u(t) + iv(t)$. For f to be analytic, we require that $\frac{df}{dt}$ exists and is independent of direction. Parameterizing the curve by arc length s, we have $\frac{df}{ds} = \frac{du}{ds} + i\frac{dv}{ds}$. The Cauchy-Riemann conditions in the tangent-normal coordinates become:

$$\frac{\partial u}{\partial s} = \frac{\partial v}{\partial n}$$
$$\frac{\partial u}{\partial n} = -\frac{\partial v}{\partial s}$$

where $\frac{\partial}{\partial n}$ denotes the derivative in the normal direction. But for a curve embedded by the Hilbert transform, the normal direction corresponds precisely to variations in the delay parameter τ, and the conditions above reduce to (9).

The Hilbert embedding $\Phi_{\mathcal{H}}$ yields an analytic signal if and only if the embedding is *conformal*—it preserves angles between the tangent and normal directions.

This gives us a profound reinterpretation: **analytic functions are those whose Hilbert embeddings are locally conformal maps from the time axis into $\mathbb{R}^2$**.

The Imaginary Unit as Delay Operator

We now demystify the "imaginary" unit by interpreting it as an operator on function spaces.

[Delay Operator] For a function $x(t)$ and a delay τ, define the delay operator D_τ by:

$$(D_\tau x)(t) = x(t - \tau) \tag{10}$$

[Hilbert Operator] Define the Hilbert operator H by:

$$(Hx)(t) = \mathcal{H}[x](t) \tag{11}$$

[Hilbert Operator as Phase Shifter] For a pure sinusoid $x(t) = e^{i\omega t}$ with $\omega > 0$, the Hilbert operator acts as:

$$Hx = -ix \quad \text{(in complex notation)} \tag{12}$$

That is, H introduces a phase shift of $-\pi/2$ (equivalently, a time delay of $\pi/(2\omega)$).

The Hilbert transform of $\cos(\omega t)$ is $\sin(\omega t)$, and of $\sin(\omega t)$ is $-\cos(\omega t)$. In complex notation, $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$. Applying the Hilbert transform to the real and imaginary parts separately yields:

$$\mathcal{H}[\cos(\omega t)] = \sin(\omega t)$$
$$\mathcal{H}[\sin(\omega t)] = -\cos(\omega t)$$

Thus $\mathcal{H}[e^{i\omega t}] = \sin(\omega t) - i\cos(\omega t) = -i(\cos(\omega t) + i\sin(\omega t)) = -ie^{i\omega t}$.

The Hilbert operator satisfies $H^2 = -I$, where I is the identity operator. In this sense, H is a square root of negative identity—exactly the role played by i in complex arithmetic.

This is the key insight: **i is not a mystical "imaginary" number but the abstract representation of the Hilbert operator, which advances signals by a quarter-period delay**. The property $i^2 = -1$ is simply the statement that two successive quarter-period delays equal a half-period delay, which inverts the signal.

[Optimality of the Hilbert Delay] Among all linear operators T_τ that can be written as weighted averages of delay operators:

$$T_\tau[x](t) = \int_0^\infty w(\tau)x(t-\tau)d\tau \tag{13}$$

the Hilbert operator is the unique (up to scaling) operator that:

1. Preserves the L^2 norm of all signals
2. Is orthogonal to the identity operator: $\langle Ix, Hx\rangle = 0$ for all x
3. Satisfies the Bedrosian theorem conditions for amplitude-frequency separation

The uniqueness follows from the fact that the Hilbert transform is the only singular integral operator that is both unitary and anti-self-adjoint on $L^2(\mathbb{R})$. The orthogonality condition corresponds to the fact that a signal and its Hilbert transform are uncorrelated at zero lag. The Bedrosian theorem conditions ensure that $H(a(t)b(t)) = a(t)Hb(t)$ when the spectra of a and b are disjoint with b bandlimited to higher frequencies.

Fundamental Theorems Reinterpreted

We now show how classical results of complex analysis acquire new meaning in this dynamical framework.

Cauchy-Riemann Equations as Dynamical Constraints

[Takens-Cauchy-Riemann Theorem] Let $x(t)$ be a real-analytic function and let $\Phi_\tau[x](t) = (x(t), x(t-\tau))$ be its 2-dimensional delay embedding. This embedding yields an analytic curve (i.e., one that can be represented as the image of an analytic function) if and only if there exists a function $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that:

$$\frac{\partial F_1}{\partial x(t)} = \frac{\partial F_2}{\partial x(t-\tau)}, \quad \frac{\partial F_1}{\partial x(t-\tau)} = -\frac{\partial F_2}{\partial x(t)} \tag{14}$$

where $F(u, v) = (F_1(u, v), F_2(u, v))$ maps the embedding coordinates to a conformal representation.

This theorem says that for the delay embedding to be "nice" (analytic), the dynamics must satisfy a kind of generalized Cauchy-Riemann condition in the embedding space.

Cauchy Integral Formula as Prediction

[Cauchy Integral Formula as Takens Prediction] Let Γ be a simple closed curve in the complex plane that encloses a point z_0, and let f be analytic inside and on Γ. In the Takens embedding picture, Γ corresponds to a closed trajectory in the reconstructed phase space. The Cauchy integral formula:

$$f(z_0) = \frac{1}{2\pi i} \oint_\Gamma \frac{f(z)}{z - z_0} dz \tag{15}$$

represents the fact that knowledge of the observable f along a complete cycle Γ determines its value at any interior point z_0—a statement about the determinism of the underlying dynamical system.

[Interpretation] In phase space reconstruction, points on Γ correspond to states visited by the system. The Cauchy kernel $(z - z_0)^{-1}$ acts as a weighting function that averages information from all points on the cycle to predict the state at z_0. This is analogous to how delay-coordinate embeddings use past observations to predict future states.

Riemann Mapping Theorem as Normal Form

[Riemann Mapping Theorem as Dynamical Normal Form] Any simply connected proper subset $U \subset \mathbb{C}$ can be conformally mapped to the unit disk $\mathbb{D}$. In dynamical systems terms, this means that any sufficiently nice attractor (with appropriate topology) can be smoothly deformed into the simplest possible shape—the unit disk—while preserving the analytic structure of the dynamics. This is exactly analogous to finding a coordinate transformation that linearizes a nonlinear system around a fixed point.

The unit disk corresponds in filter theory to the region of stability: poles inside the unit circle give stable discrete-time systems. The Riemann mapping theorem thus guarantees that any simply connected region can be mapped to this canonical stability region.

Bedrosian's Theorem and Time-Frequency Separation

The Bedrosian theorem is crucial for understanding when the Hilbert embedding yields a clean amplitude-phase decomposition.

[Bedrosian, 1963] Let $a(t)$ and $b(t)$ be functions whose Fourier spectra are supported on disjoint frequency intervals, with the spectrum of $a(t)$ confined to lower frequencies than that of $b(t)$. Then:

$$\mathcal{H}[a(t)b(t)] = a(t)\mathcal{H}[b(t)] \tag{16}$$

[Takens-Bedrosian Condition] In the delay embedding picture, the Bedrosian theorem states that if the dynamics separate into slow amplitude variations and fast oscillations, then the Hilbert embedding factors:

$$\Phi_{\mathcal{H}}[a(t)b(t)] = (a(t)b(t), a(t)\tilde{b}(t)) \tag{17}$$

where $\tilde{b}(t) = \mathcal{H}[b(t)]$. This means the amplitude $a(t)$ modulates the entire embedding, preserving the circular structure of the fast oscillation.

This explains why the analytic signal representation $z(t) = a(t)e^{i\phi(t)}$ works so well for signals with well-separated time scales—the embedding cleanly separates amplitude from phase.

Extension to Higher Dimensions and Nonlinear Systems

Our framework naturally extends beyond linear analytic functions to more general dynamical systems.

The Koopman Operator Connection

The Koopman operator provides a linear perspective on nonlinear dynamics.

[Koopman Operator] For a dynamical system $\phi_t : M \to M$, the Koopman operator $\mathcal{K}_t$ acts on observables $g : M \to \mathbb{C}$ by:

$$(\mathcal{K}_t g)(x) = g(\phi_t(x)) \tag{18}$$

[Analytic Eigenfunctions as Optimal Embeddings] Eigenfunctions of the Koopman operator satisfy $\mathcal{K}_t\psi = e^{\lambda t}\psi$ for some $\lambda \in \mathbb{C}$. When $\lambda = i\omega$, these eigenfunctions yield analytic signals:

$$\psi(x(t)) = \psi(x(0))e^{i\omega t} \tag{19}$$

The real and imaginary parts of ψ are Hilbert transform pairs along trajectories.

This shows that **Koopman eigenfunctions provide the natural coordinates for analytic embeddings of nonlinear systems**.

Synchrosqueezing as Adaptive Delay Selection

The synchrosqueezing transform reassigns energy in the time-frequency plane based on the phase derivative of the analytic signal:

$$\omega(a, b) = \frac{\partial}{\partial b} \arg(W_f(a, b)) \tag{20}$$

where $W_f(a, b)$ is the continuous wavelet transform. This can be interpreted as adaptively selecting the local optimal delay based on the instantaneous frequency, generalizing the fixed quarter-period delay of the Hilbert transform.

Discussion and Philosophical Implications

Our development leads to several profound conclusions:

1. **Complex numbers are not "imaginary"**: They are the natural coordinates for representing oscillatory phenomena with a quarter-period delay between coordinates.

2. **Analyticity is an embedding property**: A function is analytic precisely when its delay embedding is conformal and information-preserving.

3. **Cauchy-Riemann equations are dynamical constraints**: They ensure that the embedding preserves local angles and orientation, which is essential for reconstructing dynamics without distortion.

4. **The Hilbert transform is nature's optimal delay**: Among all possible linear delay embeddings, the Hilbert transform uniquely preserves energy, orthogonality, and time-frequency separation.

5. **Complex analysis is the linear theory of oscillatory dynamical systems**: Just as Fourier analysis handles linear time-invariant systems, complex analysis provides the mathematical framework for understanding systems with a well-defined analytic embedding.

This perspective suggests a research program for extending complex analytic methods to nonlinear and non-stationary systems through adaptive embeddings and Koopman operator theory.

Conclusion

We have demonstrated that complex analysis can be fruitfully reinterpreted as the study of dynamical systems under the specific, optimal delay embedding provided by the Hilbert transform. This demystifies the traditional presentation of complex numbers and analytic functions, grounding them in the concrete language of dynamical systems and signal processing.

The "imaginary" unit i emerges not as a mystical quantity but as the operator representing a quarter-period delay—the optimal lag for preserving oscillatory information. The Cauchy-Riemann equations become conditions for the embedding to be conformal. The fundamental theorems of complex analysis transform into statements about determinism, prediction, and normal forms in dynamical systems.

This unified view opens new avenues for extending classical complex analytic methods to nonlinear, non-stationary, and high-dimensional systems through the tools of modern dynamical systems theory.

Bibliography

[1] Takens, F. (1981). Detecting strange attractors in turbulence. In *Dynamical Systems and Turbulence*, Springer Lecture Notes in Mathematics, 898, 366-381.

[2] Hilbert, D. (1912). Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen. *Teubner, Leipzig.*

[3] Bedrosian, E. (1963). A product theorem for Hilbert transforms. *Proceedings of the IEEE*, 51(5), 868-869.

[4] Koopman, B. O. (1931). Hamiltonian systems and transformation in Hilbert space. *Proceedings of the National Academy of Sciences*, 17(5), 315-318.

[5] Gabor, D. (1946). Theory of communication. *Journal of the Institution of Electrical Engineers*, 93(26), 429-457.

[6] Ville, J. (1948). Théorie et applications de la notion de signal analytique. *Câbles et Transmission*, 2A, 61-74.

[7] Packard, N. H., Crutchfield, J. P., Farmer, J. D., & Shaw, R. S. (1980). Geometry from a time series. *Physical Review Letters*, 45(9), 712.

[8] Sauer, T., Yorke, J. A., & Casdagli, M. (1991). Embedology. *Journal of Statistical Physics*, 65(3-4), 579-616.

[9] Huang, N. E., et al. (1998). The empirical mode decomposition and the Hilbert spectrum for nonlinear and non-stationary time series analysis. *Proceedings of the Royal Society A*, 454, 903-995.

[10] Daubechies, I., Lu, J., & Wu, H. T. (2011). Synchrosqueezed wavelet transforms: An empirical mode decomposition-like tool. *Applied and Computational Harmonic Analysis*, 30(2), 243-261.

[11] Mees, A. I., & Judd, K. (1993). Dangers of geometric filtering. *Physica D*, 68(3-4), 427-436.

[12] Kennel, M. B., Brown, R., & Abarbanel, H. D. (1992). Determining embedding dimension for phase-space reconstruction using a geometrical construction. *Physical Review A*, 45(6), 3403.

Mathematical Background

Properties of the Hilbert Transform

The Hilbert transform has several key properties used throughout this paper:

1. **Linearity:** $\mathcal{H}[af + bg] = a\mathcal{H}[f] + b\mathcal{H}[g]$
2. **Anti-self-adjointness:** $\int_{\mathbb{R}} f\mathcal{H}[g]dt = -\int_{\mathbb{R}} \mathcal{H}[f]gdt$
3. **Unitarity:** $\|\mathcal{H}[f]\|_{L^2} = \|f\|_{L^2}$
4. **Square root of -I:** $\mathcal{H}^2[f] = -f$
5. **Commutation with derivatives:** $\mathcal{H}[f'] = (\mathcal{H}[f])'$

Takens' Theorem Conditions

The full statement of Takens' theorem requires:

1. M is a compact manifold of dimension m
2. $\phi : M \to M$ is a C^2 diffeomorphism
3. $h : M \to \mathbb{R}$ is a C^2 function
4. The set of periodic points of period $\leq 2m$ is finite

Under these conditions, the map $\Phi_{(\phi,h)} : M \to \mathbb{R}^{2m+1}$ is an embedding for generic (h, ϕ).

Connection to Filter Theory

In filter theory, the transfer function $H(s) = \frac{1}{sRC+1}$ corresponds to a pole at $s = -1/RC$. In the time domain, this gives impulse response $h(t) = e^{-t/RC}u(t)$. The analytic signal of this impulse response is:

$$z(t) = e^{-t/RC}u(t) + i\mathcal{H}[e^{-t/RC}u(t)](t) \tag{21}$$

whose trajectory in the complex plane is a spiral converging to the origin—the phase portrait of the dynamical system.