The Attralucian Essays:

Exploring the Finite

First Edition

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Arithmetic from Finite Density: A Geofinitist Foundation Based Solely on Measurable Bounds

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Historical Introduction: Where Every Previous Programme Stopped

In 1925 David Hilbert declared that "no one shall expel us from the paradise that Cantor has created." Six years later Kurt Gödel proved that, if we stay inside any sufficiently strong formal system, we are already expelled: there will always be true arithmetic statements that the system cannot prove.

The responses split into three broad camps:

Platonism / classical mathematics: accept completed infinity, accept non-constructive existence, live with Gödelian incompleteness.

Logicism / formalism (Russell–Whitehead, Curry, early Turing): attempt to reduce mathematics to pure syntax or set theory, but every attempt either reintroduces impredicative definitions or requires an Axiom of Infinity that is not finitarily justifiable.

Strict finitism / ultrafinitism (Esanin-Volpin, Pavlovic, Nelson, Lavine, Friedman): reject infinity entirely, but retain symbols as zero-volume abstract marks on an unspecified or open container. Arithmetic is preserved by fiat ("we only perform finitely many operations") without ever explaining why the physical marks themselves do not overflow a closed system.

All three camps share one unspoken agreement: the physical carrier of symbols has no intrinsic volume and no intrinsic geometry. Symbols are treated as dimensionless points that can be placed on an ever-expanding tape, page, or blackboard.

This unspoken agreement is empirically false. Every symbol that has ever existed has occupied positive, finite, measurable volume—whether in ink, phosphor, magnetic domains, synaptic vesicles, or holographic storage. The present essay removes that agreement and examines what remains of mathematics when symbols are required to pay rent in actual space.

Minimal Geofinite Postulates

We require only four postulates. Each is a direct report of existing measurement, not a philosophical stipulation.

Postulate 1 (Physicality)
Every symbol that actually exists is a physical configuration of matter–energy.

Postulate 2 (Finite Minimum Volume)
There exists a smallest length scale ℓ_0 at which differences in physical configuration can be reliably distinguished and communicated between observers using current or foreseeable technology.
As of November 2025 the tightest shared bound is the reduced Planck volume

$v_0 \approx 4.22 \times 10^{-105}$ m^3
(derived from $\ell_P = \sqrt{\hbar G/c^3} \approx 1.616 \times 10^{-35}$ m). This bound has been tightened by fourteen orders of magnitude since 1998 (from proton-scale proposals) and continues to fall only when new storage physics is publicly demonstrated.

Postulate 3 (Closure)
The observable universe is finite. Its information-carrying capacity is therefore bounded above by $\approx 10^{120}$–10^{123} bits (Bekenstein–Bousso covariant entropy bound). For any concrete computational or cognitive process we may treat the usable local container as finite.

Postulate 4 (Distinguishability)
Two symbols are distinct if and only if at least one elementary volume unit v_0 differs in state between them.

These four statements are not contestable in principle; they are contestable only by performing a measurement that violates one of them. Until such a measurement is publicly reproduced, they remain the factual background against which mathematics is practised.

Core Definitions

Nexil The nexil is the smallest distinguishable physical configuration allowed by Postulates 2 and 4. Its effective volume is v_0. A nexil is indivisible by definition: there is no public procedure that splits it while preserving distinguishability.

Alphon An alphon $A(\alpha)$ is a fixed, finite alphabet of α distinct nexils $\{s_1, s_2, \ldots, s_\alpha\}$. The choice of α is arbitrary but fixed for a given representational system ($\alpha = 2$ for binary electronics, $\alpha = 10$ for human decimal practice, $\alpha = 95$ for ASCII text, etc.).

The Null Symbol (0_s) A dedicated symbol within every alphon representing the *absence of a mark* or an *unoccupied state*. It is a placeholder, not a countable object. The symbol $\infty|0$ (a zero crossed by the finity symbol) may be used to visually emphasize its role as a geometric void rather than a positive quantity.

Alphonic Space The alphonic space is the complete set of all possible finite strings over $A(\alpha)$ that can be physically instantiated inside a chosen bounded container.

Alphonic Limit (AL) The Alphonic Limit is the smallest measurable spatial radius within which a single nexil can be reliably distinguished—effectively the spherical packing radius for a single symbol. This is a fundamental lower bound on resolution.

Alphonic Maximum ($N_{\max}$) The maximum number of occupied nexil sites that can be packed into a given container of volume V. This is determined by the Alphonic Limit and the packing geometry:

$$N_{\max} = \lfloor V/AL^3 \rfloor$$

This number is finite and fixed for any concrete physical substrate.

Density of a representation For any string s representing a mathematical object x, the density $\rho(x)$ is

$$\rho(x) = \frac{\text{number of occupied nexils in } s}{N_{\max}}$$

Density Addition and the Origin of Arithmetic

Consider two objects a and b represented by strings s_a and s_b inside the same alphonic space.

Definition of physical addition
Place s_a and s_b into the same container V without erasing either. The only physical operations permitted are
(i) local rearrangement of nexils,
(ii) deterministic overwrite rules (carry rules) that are fixed in advance and identical for every observer.

Result
The container now contains more occupied nexils than it did before. Because $N_{\max}$ is fixed, the only way to reach a new stable, single-layer configuration is to allow carry rules to propagate until the total occupied count settles to a new equilibrium distribution.

Theorem (Density Addition)

There exists exactly one stable, observer-independent final distribution that
(a) uses no more than $N_{\max}$ nexils,
(b) preserves the distinguishability of the original separate identities of a and b,
(c) minimises unused capacity (or, equivalently, maximises average local density).

That unique final distribution is the string we have conventionally named "$a + b$".

Example
In any alphon where the string "2" occupies 3 nexils and the carry table is the usual schoolbook one, merging two copies of "2" forces a carry chain that terminates in a 3- or 4-nexil string conventionally named "4". No other stable configuration is possible without violating the alphonic limit or distinguishability.

Corollary
The statement $2 + 2 = 4$ is not a Platonic truth, nor a logical truth, nor an analytic truth. It is the report of a physical relaxation process inside a finite container with indivisible unit symbols.

The same argument applies mutatis mutandis to any arithmetic operation that can be executed concretely.

Addition: The Abacus as Archetype
This process is not a mere analogy. A classical abacus is a direct, macroscopic-scale instantiation of the entire

geofinitist system. The beads are the nexils; the rods and frame define the alphonic space; the act of adding numbers is the physical merging of bead configurations; and the carry rule is a literal, mechanical procedure for managing density overflow. The abacus does not model arithmetic; it performs it. The truth of $2 + 2 = 4$ is the observed, final state of the apparatus after the physical operation is complete. Every pocket calculator, every GPU, every enzyme doing phosphorylation arithmetic in a cell, is just a very small, very fast abacus made of electrons or molecules instead of wooden beads. All concrete mathematics shares this nature.

Discussion

Explicit Empirical Claims Open to Contest

The entire argument rests on precisely three contestable empirical claims:

Empirical Claim 1. There exists a smallest distinguishable volume $v_0 > 0$.

Empirical Claim 2. The observable universe (or any local substrate we actually use) has finite information capacity.

Empirical Claim 3. Arithmetic operations are required to be publicly reproducible by physical rearrangement plus fixed overwrite rules.

If any experimenter publicly demonstrates a reliable, shared storage or distinction of information at a scale smaller than the current v_0, then Claim 1 is falsified and the bound $N_{\max}$ moves downward. The arithmetic truths remain unchanged; only the numerical value of the density ceiling changes.

If cosmology or quantum gravity ever demonstrates an actually infinite information capacity in a finite spatial volume, Claim 2 fails and classical mathematics regains its footing.

If a community agrees to accept non-physical, non-reproducible symbol manipulation (e.g. oracles, completed infinities, impredicative definitions), then Claim 3 is abandoned and we are back in the classical paradise Hilbert wanted.

Until such demonstrations are made and reproduced, the three claims remain the measured facts on the ground.

Addition: The Physical Origin of Logic

A common objection is that the “carry rules” are themselves abstract logical axioms smuggled in. This is a misunderstanding. These rules are not abstract choices; they are the simplest, most stable geometric movements for managing density in a constrained space. They are discovered, not invented. The “laws of logic” (e.g., a nexil cannot be in two states at once) are direct consequences of the geometry of distinguishability (Postulate

4). A system with different, more complex rules is physically possible but would be less efficient, requiring “extra lines in the algorithm”—meaning more energy, more time, and more complex mechanisms. Standard arithmetic and logic are, in this view, the paths of least resistance for information processing in a finite geometric universe. They are the physics of symbol manipulation.

Consequences for Physics (Conjectural but Precise)

Energy cost of arithmetic, maximum computational density, arrow of time, and quantum-gravity interface all receive crisp predictions (detailed in the original draft).

Relation to Existing Programmes

Compared with strict finitism: we do not merely restrict the height of proofs; we give the physical reason why height must be restricted.

Compared with physics-based programmes (Deutsch, Tegmark MUH): we do not derive physics from mathematics; we derive mathematics from the measured properties of the physical carrier.

Compared with category-theoretic or type-theoretic foundations: we require every morphism to pay a nexil toll.

Invitation to Criticism

The proofs in Section IV depend on no hidden axioms of set theory, logic, or infinity. They depend only on the three measurement-based claims above. Critics are invited to contest those claims with experiment, not with alternative philosophies. If no such experiment is forthcoming, then — for any mathematics that claims to describe performed calculations in the actual universe — the density argument is complete.

Conclusion

Arithmetic works because symbols have positive volume, containers are finite, and addition is physical merging followed by relaxation to the unique densest stable state. The abacus is not a model of math; it is math. The laws of logic are the observed laws of geometric movement for distinguishable symbols. Everything else is notation, or imagination. Until the day someone demonstrates a single distinguishable mark occupying less than one Planck volume, or an actually infinite container, arithmetic remains a physical science, not a branch of theology.

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