The Attralucian Essays:

Exploring the Finite

First Edition

The Attralucian Essays

From Napier's Bones to Nexils: Logarithms in Finite Symbolic Mechanics

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Chapter 1

From Napier's Bones to Nexils: Logarithms in Finite Symbolic Mechanics

The Greatest Labor-Saving Device Before the Computer

In 1614 a Scottish baron named John Napier published a book whose title said it all: *Mirifici Logarithmorum Canonis Descriptio* — "The Description of the Wonderful Canon of Logarithms." The short version of history credits Napier alone. The fuller story is richer. Around the same time a Swiss clockmaker, Jost Bürgi, had arrived at the same insight but published six years later. It was an English mathematician, Henry Briggs, who visited Napier in Edinburgh in 1615, saw the awkwardness of the original tables, and spent the next decade turning them into the clean base-10 logarithms we still recognize today. Together they gave the world one of the most powerful labor-saving inventions in the history of calculation — right up there with zero and positional notation. Multiplication and division, once grindingly slow, became simple addition and subtraction. Navigation, astronomy, engineering, and commerce all accelerated overnight.

Yet the deeper question remains: why did it work? What hidden machinery made the trick possible?

Classical accounts stop at the elegant formula. Finite Symbolic Mechanics pulls the curtain back and shows that the machinery was always finite, always measurable, and always 3-dimensional. The logarithm was never an abstract function living in the infinite continuum. It was a two-stage compression technology built on the same physical, bead-by-bead reality that any real abacus user already knows.

> *Note to the reader:* This essay presents one accessible model within the broader Geofinite framework. For the full formal development — including the Alphonic Limit, the Spherical Symbolic Geometry Mean (SGM), and the theorem of No Invariant Representation — see *The Attralucian Essays* (Haylett, 2025). The present work is self-contained but gains depth when read alongside that foundational text.

Napier's Core Insight — The Moving-Points "Movie"

Napier did not pull logarithms out of thin air. He spent more than twenty years building a mental movie of two particles moving along parallel lines.

- **Line A (arithmetic):** A point P moves at constant speed — 1 unit per second. After t seconds its position is simply t.

- **Line G (geometric):** A point Q starts at 1 and moves toward a fixed wall at 10^7 (Napier's choice to keep numbers manageable and avoid decimals). Its speed at any moment is proportional to its current distance from the wall.

Napier's three rules were simple and mechanical:

1. Both points start together ($t = 0$: P at 0, Q at 1).

2. Q heads toward the fixed wall at 10^7.

3. For every tiny step P takes, Q moves a distance proportional to its current speed.

Play the movie slowly and the relationship leaps out:

Time (s)	Pt.P on Line A	Pt.Q on Line G	What happened
0	0	1	Both start
1	1	2	Q doubles
2	2	4	doubles again
3	3	8	doubles again

Adding 1 on the arithmetic line corresponds exactly to multiplying by 2 on the geometric line. Napier's formal statement was elegant:

> "The logarithm of a number is the distance P has traveled while Q travels from 1 to that number."

The idea was pure: turn multiplication into addition by mapping geometric growth to linear motion.

The Classical "Leak"

Briggs' visit in 1615 fixed the practical problems. He convinced Napier to shift the target so that $\log(1) = 0$ and $\log(10) = 1$. The result was the familiar common logarithm (base 10) that powered science for three centuries.

Yet even Briggs' version still rested on two hidden assumptions that Finite Symbolic Mechanics refuses:

- The lines are infinitely long and infinitely divisible.

- The moving point Q can slow down through infinitely many infinitesimal steps.

In other words, classical logarithms leak infinity. They assume a continuum that no finite measurement can ever reach. They hide the actual labor of construction behind smooth notation and limits.

Napier himself knew the cost: he spent twenty years calculating the first tables by brute-force iteration because calculus had not yet been invented. Every entry was a finite, laborious step — exactly the kind of work Finite Symbolic Mechanics insists must remain visible.

Finite Symbolic Mechanics — The Grounding

Finite Symbolic Mechanics begins with a single, uncompromising commitment: all measurement is finite. There are no completed infinities, no actual limits at infinity, and no abstract real numbers floating free of measurement.

The prime epistemological limit is the **Alphonic Limit** — the isotropic bound on first-order/direct measurement at this epoch. Because we cannot resolve any direction more accurately than any other, every interactional identity must be modeled as being *contained within* a sphere.

That sphere is the **containment volume of the Nexil**.

Let us be precise:

- A **Nexil** is a *distinction event* — the smallest measurable difference that can be reliably detected and later retrieved. A Nexil has no intrinsic size, shape, or volume. It is pure relational addressability: a "mark" in the most abstract sense.

- The **Alphonic Limit** V_α is the *spherical containment volume* within which a Nexil must reside to remain measurable. Its radius r_α is the smallest resolvable radius of distinction in physical measurement.

- Because no orientation can be resolved within that region, the only rational geometry for this containment volume is spherical:

$$V_\alpha = \frac{4}{3}\pi r_\alpha^3.$$

  This sphere is not the Nexil. It is the *uncertainty envelope* of measurement itself.

Numbers are not abstract points on a line. They are **numerical alphons** — aggregates of Nexils arranged on rods exactly as beads sit on a real physical abacus. The radix 10 is not arbitrary; it emerges from the place-value mechanics of the abacus itself. When ten Nexils accumulate on one rod, they *collapse* into one Nexil on the next rod to the left — a clean, visible carry operation.

Linear arithmetic is simply Nexil counting. The deeper insight — the mapping from linear to geometric (cubic) — appears the moment we consider the 3D volumetric reality of the *containment spheres* themselves. Each Nexil occupies (or rather, is contained within) a spherical volume; when place-value rods stack or when we measure physical quantities in 3D space, the geometry of packed spheres becomes cubic. That cubic scaling is not an add-on; it is forced by the spherical containment required by the Alphonic Limit.

Logarithms Reconstructed as Two-Stage Abacus Mechanics – With a Worked Example

In Finite Symbolic Mechanics the logarithm is no longer a mysterious continuous function. It is a two-stage compression device that lives entirely inside the abacus.

Stage 1 — Table Lookup (the pre-computed map)

Napier's twenty-year labor was the construction of a finite table of Nexil configurations. He did this by iterating the moving-point rule over and over:

1. Start with Q at 1 and P at 0.

2. For each tiny time step, move P forward by a fixed amount.

3. Move Q forward by an amount equal to (its current distance to the wall at 10^7) multiplied by a tiny constant factor.

4. Record the new position of P opposite the new position of Q.

5. Repeat thousands upon thousands of times.

Each entry in the finished table is simply a frozen snapshot: "the number of Nexil-shifts on the arithmetic rod that corresponds to this geometric number on the other rod." The table itself is an alphon — a structured collection of rods holding these pre-calculated values.

Stage 2 — Addition on the abacus

To multiply two numbers a and b:

- Look up the pre-computed arithmetic-Nexil count for a.

- Look up the pre-computed arithmetic-Nexil count for b.

- Add the two arithmetic alphons with ordinary Nexil shifts and carries.

- Look up the resulting total Nexil count in the antilog direction to read off the product.

Every operation remains visible, mechanical, and finite. Multiplication has been reduced to addition of logs exactly as Napier dreamed — but now the mechanism is restored in 3D space with measurable containment spheres. No infinite lines, no infinitesimal speeds, no hidden calculus.

Worked Example – Reconstructing Napier's Logarithm Step by Step

We now take the mechanical process and make it concrete. Every step is shown exactly as Napier performed it — by hand, with finite iterations — then translated onto the real abacus of Finite Symbolic Mechanics.

Step 1: Set up the two-line mechanical model

- Line A (arithmetic): Point P moves at constant speed of 1 unit per second.
- Line G (geometric): Point Q starts at 1 and moves toward a fixed wall at 10^7. Its speed at any moment is proportional to its current distance from the wall.

Both start together at $t = 0$: $P = 0$, $Q = 1$.

Step 2: The synchronization rule (Napier's actual method) For every tiny time step Δt:

- Move P forward by Δt (fixed amount).

- Move Q forward by: (current distance of Q from the wall) $\times$ (tiny constant factor).

Napier chose the constant and the wall position (10^7) so the numbers stayed manageable. He repeated this rule thousands of times, updating Q's position after each step and recording the matching P value. No calculus — just repeated finite arithmetic.

Step 3: Tiny concrete iteration (the first few steps Napier would have done by hand)

Step	Arith.P	Geom.Q	Dist.to wall ($10^7 - Q$)	How Q moved
0	0	1	9 999 999	|
1	1	2	9 999 998	+1
2	2	4	9 999 996	+2
3	3	8	9 999 992	+4

Note: These numbers illustrate the doubling pattern. Napier's actual table required thousands of iterations between each shown row; the pattern here compresses the labour to reveal the underlying relationship.

You can see the pattern emerge immediately: each time P increases by 1, Q roughly doubles. Napier continued this process until he had filled the entire table. Every single entry was built this way: finite, step-by-step arithmetic.

Step 4: Translate the entire process into Finite Symbolic Mechanics (the real abacus)

- Each "unit" on Line A is now a Nexil (a distinction event) contained within its spherical envelope of radius r_α.
- The geometric position of Q is the number of Nexils on a rod (an alphon).
- The "wall at 10^7" is the place-value carry threshold (10 Nexils collapse into 1 on the next rod).

To build the log table on the abacus (exactly what Napier did, now visible):

1. Start with an empty arithmetic rod ($P = 0$) and a units rod with 1 Nexil ($Q = 1$).
2. For each step: slide 1 Nexil onto the arithmetic rod; calculate and slide the proportional Nexils onto the geometric rod; record the pair.
3. When any rod reaches 10 Nexils, perform the carry.

The finished table is simply a set of paired alphons: "this many arithmetic Nexils corresponds to this geometric number."

Step 5: To multiply $2 \times 4 = 8$ using the table:

1. Look up the arithmetic-Nexil count for 2.
2. Look up the arithmetic-Nexil count for 4.
3. Add those two arithmetic alphons Nexil-by-Nexil (with carries).

4. The resulting total Nexil count on the arithmetic rods is the log of 8.

5. Look up that total in the table's antilog direction $\rightarrow$ read off the original number 8.

Every Nexil is a measurable distinction event. The Alphonic Limit sets the smallest resolvable distance (the radius r_α of the containment sphere). The entire multiplication is now a visible, finite, 3-dimensional process — exactly the labor Napier performed, but now made transparent.

Two Possible Geometries of the Carry (FSM Completeness)

We have established that every Nexil is *contained within* a spherical uncertainty envelope of radius r_α, the Alphonic Limit. But what *happens* when ten Nexils on a lower rod are carried to one Nexil on the next higher rod?

The Alphonic Limit fixes the *smallest* resolvable volume. It does *not* tell us whether the higher Nexil occupies a larger sphere, the same sphere, or something else entirely. Multiple finite geometries remain possible. Here are two.

Model A: Volume Conservation (The "Compression" Model) Assume that the total *containment volume* of the ten lower Nexils is conserved during the carry.

Let each lower Nexil have containment sphere volume $V_\alpha = \frac{4}{3}\pi r_\alpha^3$. Then:

$$\text{Total volume before carry} = 10V_\alpha$$

After the carry, this volume is *re-packed* into the containment sphere of a single higher Nexil with radius r_1:

$$\frac{4}{3}\pi r_1^3 = 10 \cdot \frac{4}{3}\pi r_\alpha^3$$
$$r_1 = \sqrt[3]{10}\, r_\alpha \approx 2.154\, r_\alpha$$

Interpretation: Higher-place Nexils are physically larger. The abacus rod is a *scaling hierarchy* of spheres. The logarithm counts how many times the containment radius has been multiplied by $\sqrt[3]{10}$.

Useful for: Modeling physical substrates where distinguishability requires *spatial separation* that scales with place value (e.g., optical registers, mechanical abaci with physically larger beads on higher rails).

Model B: Fixed Radius, Variable Meaning (The "Reference" Model) Assume instead that *all* Nexils — regardless of rod position — have the *same* containment radius r_α. The carry does *not* change the sphere size. It changes the *provenance* of the Nexil.

One higher Nexil *stands for* ten lower Nexils. Its containment sphere is no larger, but its *meaning flux* ΔM — the work required to maintain its distinctness — is greater because it must *refer to* a packed set of lower distinctions.

In this model, the larger sphere ($\sqrt[3]{10}\, r_\alpha$) is *not* occupied by any Nexil. It is a *mathematical abstraction*: the sphere that *would be* required if we tried to contain ten lower Nexils without carrying. The carry *avoids* needing that larger sphere. That is the whole efficiency of place-value notation.

Interpretation: Higher-place Nexils are not larger. They are *denser in meaning*. The logarithm counts how many layers of reference have been folded into a single Nexil.

Useful for: Modeling symbolic systems where the *medium* is uniform (e.g., electronic memory, paper, quantum dots) but the *interpretation* is layered.

Model C (and beyond) These two are not exhaustive. Other models are possible:

- **Mixed models:** Radius scales only after a threshold number of carries (e.g., every third carry doubles the radius).
- **Substrate-dependent models:** Silicon (fixed r_α) vs. optical (scaling r_α) vs. biological (adaptive r_α).
- **Observer-relative models:** The same physical configuration yields different containment geome-

tries for different measuring instruments.

The Alphonic Limit constrains *all* models. It does *not* select one.

What This Means for the Logarithm In **Model A**, the logarithm is a *physical compression count*: $\log_{10} N$ is the number of times the containment radius has been multiplied by $\sqrt[3]{10}$.

In **Model B**, the logarithm is a *reference depth count*: $\log_{10} N$ is the number of layers of meaning folded into the representation.

Both are finite. Both are geometric. Both are measurable in principle. Neither requires infinity.

The classical logarithm — the smooth, continuous, base-invariant function — is the *idealized limit* of both models as $r_\alpha \to 0$. But that limit is *never reached* in any finite measurement. Geofinitism keeps the models separate and refuses the limit.

From Napier's Tables to Silicon Registers – The Same Mechanics in Modern Computation

The beauty of Finite Symbolic Mechanics is that it does not stop at historical curiosity. It explains today's machines with the same rules — and both Model A and Model B find natural interpretations.

Modern computers do not use physical beads on rods. They use transistors — microscopic electronic switches etched onto silicon chips. Each transistor is incredibly simple: it can be only in one of two states, like a tiny light switch.

$$\text{Off} = 0 \qquad \text{On} = 1$$

That is all. One transistor = one binary digit = one bit. In Finite Symbolic Mechanics terms, a bit is the modern Nexil: a distinction event (0 or 1) contained within a physical region of size determined by the lithographic pitch. That pitch is the contemporary Alphonic Limit r_α (currently $\sim$ 3–10 nm for advanced nodes).

A register is just a row of these switches working together, exactly like one rod on a real abacus. A typical register in your laptop or phone holds 64 bits (64 tiny switches lined up). The entire row can represent a number by turning different switches on or off — the same way Nexils on an abacus rod represent a digit.

How arithmetic actually happens inside the chip (the digital abacus)

Imagine the register as an electronic rod:

- To add two numbers, the hardware slides the "1" bits around and propagates carries — exactly as you slide Nexils and carry over when a rod fills up.

- A bit shift is simply sliding every bit one position left or right on the register. Shifting left multiplies by 2; shifting right divides by 2. It is the electronic version of moving Nexils to the next higher or lower place-value rod.

Now watch the exact same two-stage logarithm process we just saw on the abacus, but now happening inside the silicon — and notice how both geometric models apply:

Stage 1 – Table lookup / pre-computed map The floating-point unit inside the CPU already contains a small, built-in log table (or a short sequence of register operations that calculate it). When you multiply two numbers, the hardware quickly looks up (or computes) the logarithm of each — using only the finite switches in the registers. This is Napier's table, but stored in transistors instead of handwritten paper.

Stage 2 – Addition on the register (the electronic abacus) The hardware adds the two log values using ordinary bit-by-bit addition and carries. Then it performs the antilog (reverse lookup) to get the final product.

Which geometric model does silicon implement? Silicon is ambiguous — and that ambiguity is instructive. At the physical layer, each transistor occupies a fixed area (roughly constant r_α). This favours **Model B** (fixed radius, variable meaning). However, the *effective* distinction between a lower bit and a higher bit in a multi-word integer involves carry propagation that *functions as if* higher bits had larger containment (Model A). The hardware itself does not decide between the models; the models are different *interpretations* of the same finite physical process.

Every single operation is still finite, visible in the hardware schematic, and bounded by the register width (the modern Alphonic Limit). If the result cannot fit in 64 bits, the hardware rounds it — exactly as an abacus user would stop when the rod is full.

A tiny concrete picture

Suppose the computer wants to multiply 2×4 (the same example we used earlier).

- In binary (the language of transistors): 2 is stored as `10` (one bit on, one off); 4 is stored as `100`.
- The log step finds their log values (pre-computed or calculated with bit shifts).
- The addition step adds those logs in the register using carry chains.
- The antilog step reads the result back as 8 (`1000`

in binary).

All of it happens in nanoseconds, but the underlying machinery is identical: finite symbols (bits), finite addition with carries, and place-value shifting — the same mechanics Napier first built by hand four centuries ago.

This is why logarithms remain at the heart of modern computation even though we rarely type "log" in everyday code. The hardware hides the two-stage process for speed, but Finite Symbolic Mechanics reveals that the process is still the same finite, mechanical reality — only now the "Nexils" are electrons flipping switches in silicon layers. The twenty-year labor has simply been accelerated by many orders of magnitude — yet the foundational commitment to finiteness has never been broken.

Why the Twenty-Year Labor Still Matters

Napier did not "discover" an eternal truth hiding in the continuum. He constructed a finite symbolic table under the same constraints every abacus user faces. The tables were built Nexil-by-Nexil, step-by-step, exactly as Finite Symbolic Mechanics demands. The classical narrative erases that labor and replaces it with a smooth curve. Geofinitism restores the labor and shows that the power of logarithms comes from the finiteness, not in spite of it.

Side-by-Side Comparison

Aspect	**Classical View**	**Finite Symbolic Mechanics (Geofinitism)**
Fundamental entity	Abstract real number	Nexil (distinction event)
Measurement precision	Arbitrarily improvable (limits)	Bounded by isotropic Alphonic Limit r_α
Geometry of representation	1D infinite line	3D spherical containment volumes
Carry operation	Placeholder abstraction	Volume conservation (Model A) or fixed-radius reference (Model B)
Model pluralism	None (one true mathematics)	Multiple finite models, distinguished by utility
Multiplication	Single abstract operation	Nexil shifts + carries on rods
Logarithm	Continuous function / limit	Two-stage table lookup + abacus addition

The Deeper Cubic Mapping

The original insight — that the linear-to-geometric mapping is actually linear-to-cubic — now locks into place. On the abacus the linear operation is Nexil addition on a single rod. When we consider the 3D reality of each *containment sphere* (radius r_α) or the stacking of place-value rods, the geometry becomes cubic. The sphere forces the cubic scaling into the mechanics itself — but crucially, the *form* of that scaling depends on which model (A, B, or other) we adopt for the carry operation.

In Model A, the cubic scaling is literal: $r_1^3 = 10r_\alpha^3$. In Model B, the cubic scaling is referential: the abstraction of a sphere of radius $\sqrt[3]{10}\, r_\alpha$ is useful even though no Nexil occupies it.

Both are *finite cubic geometries.* Both respect the Alphonic Limit.

Compression Without Concealment

Logarithms were never magic. They were the first great finite symbolic compression technology. Finite Symbolic Mechanics simply refuses to let the machinery disappear behind smooth notation and infinite limits. By grounding everything in Nexils, spherical containment, and the real abacus — and by showing that the same mechanics still drive every modern computer — we recover both the historical wonder and the actual physical mechanism.

But we do one thing more. We acknowledge that the Alphonic Limit does not dictate a single model of the carry. It only sets the boundary within which all possible finite models must live. Model A and Model B (and others) are not competitors for truth. They are *tools for different measurement contexts*.

On the Utility of Models (A Philosophical Coda)

The classical tradition assumes that a mathematical model aims at *unique correspondence* with a mind-independent reality. Geofinitism replaces this assumption with the foundational commitments and then the pragmatic and practical criterion: **utility**.

A model is useful if:

1. It respects the Alphonic Limit (no infinities, no infinitesimals).
2. It produces verifiable predictions about finite measurement outcomes.
3. It can be implemented on a finite substrate (abacus, paper, silicon).

Both Model A and Model B satisfy these conditions. Neither can be ruled out by pure reason. Which one we choose depends on the measurement apparatus, the substrate, and the question we are asking.

This pluralism is not relativism. The Alphonic Limit is fixed. The spherical containment geometry is derived. But *within* that finite geometric space, multiple packings, scalings, and interpretations remain possible. The job of the Geofinite mathematician is not to discover the One True Model. It is to *construct useful models* and to be explicit about their assumptions.

Napier built one model (his original tables). Briggs refined it. Silicon implements an implicit hybrid. The Attralucian Essays formalize the framework. This chapter has shown that the logarithm — in all its historical and computational power — can be understood without ever leaving the finite, measurable, 3-dimensional world where all genuine calculation has always taken place.

The story of logarithms is no longer about escaping into the continuum. It is about staying faithfully inside the finite — and about having the intellectual honesty to admit that, within that finitude, more than one geometry may serve.